

DISERTATION DRAFT #1

Singular Semi-Riemannian Geometry and Singular General Relativity

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Declaration of Authorship

I, Ovidiu Cristinel STOICA, declare that this thesis titled, 'Singular Semi-Riemannian Geometry and Singular General Relativity' and the work presented in it are my own. I confirm that:

- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

Abstract

This work presents the foundations of Singular Semi-Riemannian Geometry and Singular General Relativity, based on the author's research. An extension of differential geometry and of Einstein's equation to singularities is reported. Singularities of the form studied here allow a smooth extension of the Einstein field equations, including matter. This applies to the Big-Bang singularity of the FLRW solution. It applies to stationary black holes, in appropriate coordinates (since the standard coordinates are singular at singularity, hiding the smoothness of the metric). In these coordinates, charged black holes have the electromagnetic potential regular everywhere. Implications on Penrose's Weyl curvature hypothesis are presented. In addition, these singularities exhibit a (geo)metric dimensional reduction, which acts as a regulator for the quantum fields, including for quantum gravity, in the UV regime. This opens the perspective of perturbative renormalizability of quantum gravity without modifying General Relativity.

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Chapter 1

Introduction

1.1 Historical Background

We are interested in the properties of a class of smooth differentiable manifolds which have on the tangent bundle a smooth bilinear form (also named *metric*). Historically, the first such manifolds which were studied are the Euclidean plane and space, followed by the non-Euclidean geometries introduced by Lobachevsky, Gauss, and Bolyai. After Gauss extended the study of the Euclidean plane to curved surfaces, Bernhard Riemann generalized it to curved spaces with arbitrary number of dimensions [1]. Riemann hoped to give a geometric description of the physical space, in the idea that matter is in fact the effect of the curvature. The previously discovered geometries – the Euclidean and non-Euclidean ones, and Gauss’s geometry of surfaces – are all particular cases of *Riemannian geometry*. A Riemannian manifold is a differentiable manifold endowed with a symmetric, non-degenerate and positive definite bilinear form on its tangent bundle.

The necessity of studying spaces having a symmetric, non-degenerate bilinear form which is not positive definite appeared with the Theory of Relativity [2]. A differentiable manifold having on its tangent bundle a symmetric, non-degenerate bilinear form which is not necessarily positive or negative definite is named *semi-Riemannian manifold* (sometimes *pseudo-Riemannian manifold*, and in older textbooks even is called Riemannian manifold). It constitutes the mathematical foundation of General Relativity. It was thoroughly studied, and the constructions made starting from the non-degenerate metric, such as the Levi-Civita connection, the covariant derivative, the Riemann, Ricci and scalar curvatures are very similar to the Riemannian case, when the metric is positive definite. On the other hand, other properties, especially the global ones, are very different in the indefinite case. Very good references for semi-Riemannian geometry are the textbooks [3], [4], [5].

If we allow the metric to be degenerate, many difficulties occur. For this reason, advances were made slower than for the non-degenerate case, in particular directions. Manifolds endowed with degenerate metric were studied by Moisil [6], Strubecker [7–10], Vranceanu [11].

One situation when the metric can be degenerate is in the study of submanifolds of semi-Riemannian manifolds. In the Riemannian case, the submanifolds are Riemannian too. But in general the image of a smooth mapping from a differentiable manifold to a Riemannian or semi-Riemannian manifold may be singular, in particular may have degenerate metric. In the case of *varieties* the problem of finding a *resolution of its singularities* was proved to have positive answer by Hironaka [12, 13]. In the semi-Riemannian case, the metric induced on a submanifold can be degenerate, even though the larger manifold has non-degenerate metric. The properties of such submanifolds,

studied in many articles, *e.g.* in [14], [15, 16], [17, 18], were extended by Kupeli to manifolds endowed with degenerate metric of constant signature [19, 20]. The situation is much more difficult when the signature changes.

There are some situations in General Relativity when the metric becomes degenerate or changes its signature. There are cosmological models of the Universe in which the initial singularity of the Big Bang is replaced by making the metric Riemannian from the early Universe. Such models, constructed in connection to the Hartle-Hawking no-boundary approach to Quantum Cosmology, assume that the metric was Riemannian, and it changed, becoming Lorentzian, so that time emerged from a space dimension. Such a change of signature is considered to take place when traversing a hypersurface, on which the metric becomes degenerate (see [21],[22, 23],[24–26], [27], [28–33], [34–39] *etc.*).

Another situation where the metric can become degenerate was proposed by Einstein and Rosen, as a model of charged particles. They were the first to model charged particles as *wormholes*, also named *Einstein-Rosen bridges* [40].

The Einstein’s equation, as well as its Hamiltonian formulation due to Arnowitt, Deser and Misner [41], may lead to cases when the metric is degenerate. As the Penrose and Hawking *singularity theorems* show, the conditions leading to singularities are very general, applying to the matter distribution in our Universe [5, 42–46]. Therefore, it is important to know how we can deal with such singularities. Many attempts were done to solve this issue.

For example it was suggested that Ashtekar’s method of “new variables” [47–49] can be used to pass beyond the singularities, because the variable \tilde{E}_i^a – a densitized frame of vector fields – defines the metric, which can be degenerate. Unfortunately, it turned out that in this case the connection variable A_a^i may become singular *cf. e.g.* [50].

Quantum effects may play a role in *avoiding the singularities*. Loop Quantum Cosmology, by quantizing spacetime, provided a mean to avoid the Big-Bang singularity and replace it with a Big-Bounce, due to the fact that the curvature is bounded, because there is a minimum distance [51, 52]. Another possibility to avoid singularities is given in [53], within the Einstein-Cartan-Sciama-Kibble theory [54–56].

A more classical proposal to avoid the consequences of singularities was initiated by R. Penrose, with the *cosmic censorship hypothesis* [57–60]. According to the *weak cosmic censorship hypothesis*, all singularities (except the Big-Bang singularity) are hidden behind an *event horizon*, hence are not *naked*. The *strong cosmic censorship hypothesis* conjectures that the maximal extension of spacetime as a regular Lorentzian manifold is *globally hyperbolic*.

The literature in the approaches to singularities is too vast, and it would be unjust to claim to review it properly in a research work which is not a dedicated review. A great review on the problem of singularities in General Relativity is given in [61], and a more up-to-date one in [62].

1.2 Motivation for this Research

In this thesis is developed the mathematical formalism for a large class of manifolds with symmetric bilinear forms on the tangent structure, which are allowed to become degenerate. Then these results are applied to the singularities in General Relativity.

This special type of singular semi-Riemannian manifold has regular properties in what concerns

1. the Riemann curvature R_{abcd} (and not R^a_{bcd} , which is in general divergent),
2. the covariant derivative of a class of differential forms,
3. and other invariants and differential operators which in general cannot be defined properly because they require the inverse of the metric.

These properties of regularity are not valid for any type of degenerate metric. This justifies the name of *semi-regular semi-Riemannian manifolds* given to these special singular semi-Riemannian manifolds. Semi-regular metrics can also be used to give a densitized version to the Einstein equation, as well as to other formulations of General Relativity. These can be used to approach the problem of singularities in General Relativity.

The signature of the metric of a semi-regular semi-Riemannian manifold can change, but when it doesn't change, we obtain the "stationary singular semi-Riemannian manifolds" with constant signature, researched by Kupeli [19, 20]. These in turn generalize the semi-Riemannian manifolds, which generalize the Riemannian manifolds.

In General Relativity, Einstein's equation encodes the relation between the stress-energy tensor of matter, and the Ricci curvature. The singularity theorems of Penrose and Hawking [5, 42–46], show that the conditions of occurrence of singularities are quite common. Christodoulou [63] showed that these conditions are in fact more common, and then Klainerman and Rodnianski [64] proved that they are even more common. Therefore, it would be important to understand them.

There are two main situations in which singularities appear in General Relativity and Cosmology: at the Big-Bang, and in the black holes. We will see that the mathematical

apparatus developed in this thesis finds applications in both these situations. Another reason to research the Big-Bang singularity is due to its connection with the entropy, by Penrose's Weyl curvature hypothesis [59]. The properties of black hole singularities are also important in connection to other problems in Physics, for example in modeling particles, and in Quantum Gravity. These problems motivate this research, and the development of singular semi-Riemannian geometry.

As the previous section shows, much work is done on singular metrics. But to the author's knowledge, the systematic approach presented here and the results are novel, as well as the applications to General Relativity, and have no overlap with the research that is previously done.

1.3 Presentation per Chapter

The first part, [Singular semi-Riemannian geometry](#), contains the development of the geometry of metrics which can be degenerate, and have variable signature. Chapter 2 introduces the main properties of such manifolds. It contains an invariant definition of contraction between covariant indices, which works also when the metric is degenerate. With the help of the metric, it is shown that in some cases one can construct covariant derivatives, and even define a Riemann curvature tensor. All these invariants reduce to the known ones, if the metric is non-degenerate. Chapter 3 generalizes the notion of warped products to the case when the metric can become degenerate. It provides a simple way to construct examples of singular semi-Riemannian manifolds. Chapter 4 obtains Cartan's structural equations for degenerate metric.

The second part, [Singular general relativity](#), applies the mathematics developed in the first part to the singularities encountered in General Relativity. Chapter 5 introduces two equations equivalent to Einstein's equation, but which remains smooth at singularities. The first equation remains smooth at the so-called semi-regular singularities. The second of the equations applies to the more restricted case of quasi-regular singularities, which will turn out to be important in the following chapters.

Chapter 6 studies the properties of the Big-Bang singularity, with the apparatus developed so far. It shows that the Friedmann-Lemaître-Robertson-Walker spacetime is semi-regular, but also quasi-regular. It also studies some important properties of the FLRW singularity. Then, in section §6.3, a more general solution, which is not homogeneous or isotropic, is presented. It is shown that it satisfies the Weyl curvature hypothesis of Penrose.

The black hole singularities are studied in chapter 7. It is shown that the Schwarzschild singularity is semi-regularizable, by using a method inspired by that of Eddington-Finkelstein [65, 66], used to prove that the metric is regular on the event horizon. This method is also applied to make the Reissner-Nordstrom and the Kerr-Newman singularities analytic. Then, it is shown that this approach allows the construction of globally hyperbolic spacetimes with singularities.

Chapter 8 explores the possibility that quantum gravity is preturbatively renormalizable, by dimensional reduction at singularities.

Part I

Singular semi-Riemannian geometry

Chapter 2

Singular semi-Riemannian manifolds

The text in this chapter is based on author's paper [\[67\]](#).

On a Riemannian or a semi-Riemannian manifold, the metric determines invariants like the Levi-Civita connection and the Riemann curvature. If the metric becomes degenerate (as in singular semi-Riemannian geometry), these constructions no longer work, because they are based on the inverse of the metric, and on related operations like the contraction between covariant indices.

In this chapter we develop the geometry of singular semi-Riemannian manifolds. First, we introduce an invariant and canonical contraction between covariant indices, applicable even for degenerate metrics. This contraction applies to a special type of tensor fields, which are radical-annihilator in the contracted indices. Then, we use this contraction and the Koszul form to define the covariant derivative for radical-annihilator indices of covariant tensor fields, on a class of singular semi-Riemannian manifolds named radical-stationary. We use this covariant derivative to construct the Riemann curvature, and show that on a class of singular semi-Riemannian manifolds, named semi-regular, the Riemann curvature is smooth.

In section [§5.1](#) we will apply these results to construct a version of Einstein's tensor whose density of weight 2 remains smooth even in the presence of semi-regular singularities. We can thus write a densitized version of Einstein's equation, which is smooth, and which is equivalent to the standard Einstein equation if the metric is non-degenerate.

2.1 Introduction

2.1.1 Motivation and related advances

Let M be a differentiable manifold with a symmetric inner product structure, named metric, on its tangent bundle. If the metric is non-degenerate, we can construct in a canonical way a Levi-Civita connection and the Riemann, Ricci and scalar curvatures. If the metric is allowed to be degenerate (hence M is a singular semi-Riemannian manifold), some obstructions prevented so far the construction of such invariants.

Degenerate metrics are useful because they can arise in various contexts in which semi-Riemannian manifolds are used. They are encountered even in manifolds with non-degenerate (but indefinite) metric, because the metric induced on a submanifold can be degenerate. The properties of such submanifolds were studied *e.g.* in [15, 16], [17, 18].

In General Relativity, there are models or situations when the metric becomes degenerate or changes its signature. As the Penrose and Hawking *singularity theorems* [5, 42–46] show, Einstein's equation leads to singularities under very general conditions, apparently similar to the matter distribution in our Universe. Therefore, many attempts were done to deal with such singularities.

It is needed a generalization of the standard methods of semi-Riemannian Geometry, to cover the degenerate case. A degenerate metric prevents the standard constructions like covariant derivative and curvature. Manifolds endowed with degenerate metrics were studied by Moisil [6], Strubecker [7–10], Vranceanu [11]. Notable is the work of Kupeli [15, 16, 19], which is limited to the constant signature case.

2.1.2 Presentation of this chapter

The purpose of this chapter is to provide a toolbox of geometric invariants, which extend the standard constructions from semi-Riemannian geometry to the non-degenerate case, with constant or variable signature.

The first goal of this chapter is to construct canonical invariants such as the covariant derivative and Riemann curvature tensor, in the case of singular semi-Riemannian geometry. The main obstruction for this is the fact that when the metric is non-degenerate, it doesn't admit an inverse. This prohibits operations like index raising and contractions between covariant indices. This prevents the definition of a Levi-Civita connection, and by this, the construction of the curvature invariants. This chapter presents a way to

construct such invariants even if the metric is degenerate, for a class of singular semi-Riemannian manifolds which are named *semi-regular*.

The second goal is to apply, in the following chapters, the tools developed here to write a densitized version of Einstein's tensor which remains smooth in the presence of singularities, if the spacetime is semi-regular. Consequently, we can write a version of Einstein's equation which is equivalent to the standard one if the metric is non-degenerate. This allows us to extend smoothly the equations of General Relativity beyond the apparent limits imposed by the singularity theorems of Penrose and Hawking [5, 42–46].

Section §2.2 contains generalities on singular semi-Riemannian manifolds, in particular the radical bundle associated to the metric, made of the degenerate tangent vectors. In section §2.3 are studied the properties of the radical-annihilator bundle, consisting in the covectors annihilating the degenerate vectors. Tensor fields which are radical-annihilator in some of their covariant indices are introduced. On this bundle we can define a metric which is the next best thing to the inverse of the metric, and which will be used to perform contractions between covariant indices. Section §2.4 shows how we can contract covariant indices of tensor fields, so long as these indices are radical-annihilators.

Normally, the Levi-Civita connection is obtained by raising an index of the right member of the Koszul formula (named here Koszul form), operation which is not available when the metric is degenerate. Section §2.5 studies the properties of the Koszul form, which are similar to those of the Levi-Civita connection. This allows us to construct in section §2.6 a sort of covariant derivative for vector fields, and in §2.6.3 a covariant derivative for differential forms.

The notion of semi-regular semi-Riemannian manifold is defined in section §2.7 as a special type of singular semi-Riemannian manifold with variable signature on which the lower covariant derivative of any vector field, which is a 1-form, admits smooth covariant derivatives.

The Riemann curvature tensor is constructed in §2.7 with the help of the Koszul form and of the covariant derivative for differential forms introduced in section §2.6. For semi-regular semi-Riemannian manifolds, the Riemann curvature tensor is shown to be smooth, and to have the same symmetry properties as in the non-degenerate case. In addition, it is radical-annihilator in all of its indices, this allowing the construction of the Ricci and scalar curvatures. Then, in section §2.8, the Riemann curvature tensor is expressed directly in terms of the Koszul form, obtaining an useful formula. Then the

Riemann curvature is compared with a curvature tensor obtained by Kupeli by other means [19].

Section §2.9 presents two examples of semi-regular semi-Riemannian manifolds. The first is based on diagonal metrics, and the second on degenerate metrics which are conformal to non-degenerate metrics.

2.2 Singular semi-Riemannian manifolds

2.2.1 Definition of singular semi-Riemannian manifolds

Definition 2.1. (see e.g. [19], [68], p. 265 for comparison) A *singular semi-Riemannian manifold* is a pair (M, g) , where M is a differentiable manifold, and $g \in \Gamma(T^*M \odot_M T^*M)$ is a symmetric bilinear form on M , named *metric tensor* or *metric*. If the signature of g is fixed, then (M, g) is said to be with *constant signature*. If the signature of g is allowed to vary from point to point, (M, g) is said to be with *variable signature*. If g is non-degenerate, then (M, g) is named *semi-Riemannian manifold*. If g is positive definite, (M, g) is named *Riemannian manifold*.

Remark 2.2. Let (M, g) be a singular semi-Riemannian manifold and let $M_\lambda \subseteq M$ be the set of the points where the metric changes its signature. The set $M - M_\lambda$ is dense in M , and it is a union of singular semi-Riemannian manifolds with constant signature.

Example 2.1 (Singular Semi-Euclidean Spaces $\mathbb{R}^{r,s,t}$, cf. e.g. [68], p. 262). Let $r, s, t \in \mathbb{N}$, $n = r + s + t$. We define the singular semi-Euclidean space $\mathbb{R}^{r,s,t}$ by:

$$\mathbb{R}^{r,s,t} := (\mathbb{R}^n, \langle, \rangle), \quad (2.1)$$

where the metric acts on two vector fields X, Y on \mathbb{R}^n at a point p on the manifold, in the natural chart, by

$$\langle X_p, Y_p \rangle = - \sum_{i=r+1}^s X^i Y^i + \sum_{j=r+s+1}^n X^j Y^j. \quad (2.2)$$

If $r = 0$ we fall over the semi-Euclidean space $\mathbb{R}_s^n := \mathbb{R}^{0,s,t}$ (see e.g. [3], p. 58). If $s = 0$ we find the degenerate Euclidean space. If $r = s = 0$, then $t = n$ and we recover the Euclidean space \mathbb{R}^n endowed with the natural scalar product.

2.2.2 The radical of a singular semi-Riemannian manifold

Definition 2.3. (cf. e.g. [17], p. 1, [20], p. 3, and [3], p. 53) Let (V, g) be a finite dimensional inner product space, where the inner product g may be degenerate. The totally degenerate space $V_\circ := V^\perp$ is named the *radical* of V . An inner product g on a vector space V is non-degenerate if and only if $V_\circ = \{0\}$.

Definition 2.4. (see e.g. [19], p. 261, [68], p. 263) We denote by $T_\circ M$ and we call *the radical of TM* the following subset of the tangent bundle: $T_\circ M = \cup_{p \in M} (T_p M)_\circ$. We can define vector fields on M valued in $T_\circ M$, by taking those vector fields $W \in \mathfrak{X}(M)$ for which $W_p \in (T_p M)_\circ$. We denote by $\mathfrak{X}_\circ(M) \subseteq \mathfrak{X}(M)$ the set of these sections – they form a vector space over \mathbb{R} and a module over $\mathcal{F}(M)$. $T_\circ M$ is a vector bundle if and only if the signature of g is constant on all M , and in this case, $T_\circ M$ is a distribution.

Example 2.2. The radical $T_\circ \mathbb{R}^{r,s,t}$ of the singular semi-Euclidean manifold $\mathbb{R}^{r,s,t}$ in the Example 2.1 is spanned at each point p by the tangent vectors ∂_{ap} with $a \leq r$:

$$T_\circ \mathbb{R}^{r,s,t} = \bigcup_{p \in \mathbb{R}^{r,s,t}} \text{span}(\{(p, \partial_{ap}) | \partial_{ap} \in T_p \mathbb{R}^{r,s,t}, a \leq r\}). \quad (2.3)$$

The sections of $T_\circ \mathbb{R}^{r,s,t}$ are therefore given by

$$\mathfrak{X}_\circ(\mathbb{R}^{r,s,t}) = \{X \in \mathfrak{X}(\mathbb{R}^{r,s,t}) | X = \sum_{a=1}^r X^a \partial_a\}. \quad (2.4)$$

2.3 The radical-annihilator inner product space

Let (V, g) be an inner product vector space. If the inner product g is non-degenerate, it defines an isomorphism $\flat : V \rightarrow V^*$ (see e.g. [69], p. 15; [70], p. 72). If g is degenerate, \flat remains a linear morphism, but not an isomorphism. This is why we can no longer define a dual for g on V^* in the usual sense. We will see that we can still define canonically an inner product $g_\bullet \in \flat(V)^* \odot \flat(V)^*$, and use it to define contraction and index raising in a weaker sense than in the non-degenerate case. This rather elementary construction can be immediately extended to singular semi-Riemannian manifolds. It provides a tool to contract covariant indices and construct the invariants we need.

2.3.1 The radical-annihilator vector space

This section applies well-known elementary properties of linear algebra, with the purpose is to extend fundamental notions related to the non-degenerate inner product g on a

vector space V induced on the dual space V^* (cf. e.g. [71], p. 59), to the case when g is allowed to be degenerate. Let (V, g) be an inner product space over \mathbb{R} .

Definition 2.5. The inner product g defines a vector space morphism, named the *index lowering morphism* $\flat : V \rightarrow V^*$, by associating to any $u \in V$ a linear form $\flat(u) : V \rightarrow \mathbb{R}$ defined by $\flat(u)v := \langle u, v \rangle$. Alternatively, we use the notation u^\flat for $\flat(u)$. For reasons which will become apparent, we will also use the notation $u^\bullet := u^\flat$.

Remark 2.6. It is easy to see that $V_\circ = \ker \flat$, so \flat is an isomorphism if and only if g is non-degenerate.

Definition 2.7. The *radical-annihilator* vector space $V^\bullet := \text{im } \flat \subseteq V^*$ is the space of 1-forms ω which can be expressed as $\omega = u^\bullet$ for some u , and they act on V by $\omega(v) = \langle u, v \rangle$.

Obviously, in the case when g is non-degenerate, we have the identification $V^\bullet = V^*$.

Remark 2.8. In other words, V^\bullet is the annihilator of V_\circ . It follows that $\dim V^\bullet + \dim V_\circ = n$.

Remark 2.9. Any $u' \in V$ satisfying $u'^\bullet = \omega$ differs from u by $u' - u \in V_\circ$. Such 1-forms $\omega \in V^\bullet$ satisfy $\omega|_{V_\circ} = 0$.

Definition 2.10. On the vector space V^\bullet we can define a unique non-degenerate inner product g_\bullet by $g_\bullet(\omega, \tau) := \langle u, v \rangle$, where $u^\bullet = \omega$ and $v^\bullet = \tau$. We alternatively use the notation $\langle\langle \omega, \tau \rangle\rangle_\bullet = g_\bullet(\omega, \tau)$.

Proposition 2.11. The inner product g_\bullet from above is well-defined, being independent on the vectors u, v chosen to represent the 1-forms ω, τ .

Proof. If $u', v' \in V$ are other vectors satisfying $u'^\bullet = \omega$ and $v'^\bullet = \tau$, then $u' - u \in V_\circ$ and $v' - v \in V_\circ$. $\langle u', v' \rangle = \langle u, v \rangle + \langle u' - u, v \rangle + \langle u, v' - v \rangle + \langle u' - u, v' - v \rangle = \langle u, v \rangle$. \square

Proposition 2.12. The inner product g_\bullet from above is non-degenerate, and if g has the signature (r, s, t) , then the signature of g_\bullet is $(0, s, t)$.

Proof. Let's take an orthonormal basis $(e_a)_{a=1}^n$ in which the inner product is diagonal, with the first r diagonal elements being 0. We have $e_a^\bullet = 0$ for $a \in \{1, \dots, r\}$, and the 1-forms $\omega_a := e_{r+a}^\bullet$ for $a \in \{1, \dots, s+t\}$ are the generators of V^\bullet . They satisfy $\langle\langle \omega_a, \omega_b \rangle\rangle_\bullet = \langle e_{r+a}, e_{r+b} \rangle$. Therefore, $(\omega_a)_{a=1}^{s+t}$ are linear independent and the signature of g_\bullet is $(0, s, t)$. \square

Figure 2.1 illustrates the various spaces associated with a degenerate inner product space (V, g) and the inner products induced by g on them.

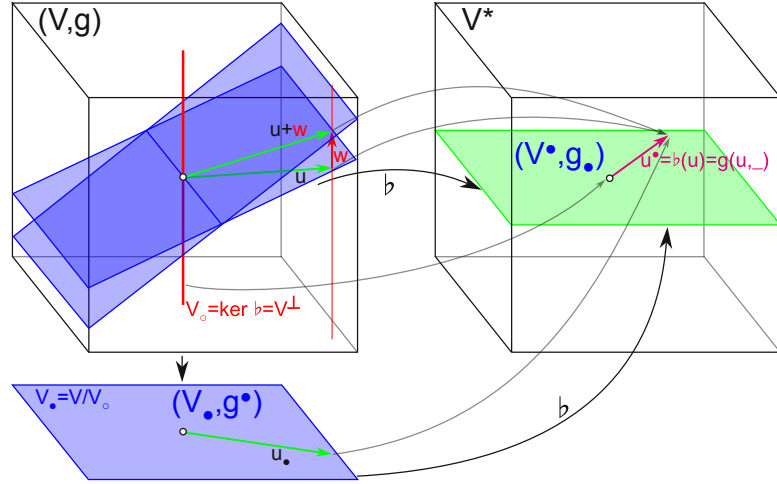


FIGURE 2.1: (V, g) is an inner product vector space. The morphism $b : V \rightarrow V^*$ is defined by $u \mapsto u^\bullet := b(u) = u^\flat = g(u, -)$. The radical $V_\circ := \ker b = V^\perp$ is the set of isotropic vectors in V . $V^\bullet := \text{im } b \leq V^*$ is the image of b . The inner product g induces on V^\bullet an inner product defined by $g_\bullet(u_1^\flat, u_2^\flat) := g(u_1, u_2)$, which is the inverse of g iff $\det g \neq 0$. The quotient $V_\bullet := V/V_\circ$ consists in the equivalence classes of the form $u + V_\circ$. On V_\bullet , g induces an inner product $g^\bullet(u_1 + V_\circ, u_2 + V_\circ) := g(u_1, u_2)$.

2.3.2 The radical-annihilator vector bundle

Definition 2.13. We denote by $T^\bullet M$ the subset of the cotangent bundle defined as

$$T^\bullet M = \bigcup_{p \in M} (T_p M)^\bullet \quad (2.5)$$

where $(T_p M)^\bullet \subseteq T_p^* M$ is the space of covectors at p which can be expressed as $\omega_p(X_p) = \langle Y_p, X_p \rangle$ for some $Y_p \in T_p M$ and any $X_p \in T_p M$. $T^\bullet M$ is a vector bundle if and only if the signature of the metric is constant. We can define sections of $T^\bullet M$ in the general case, by

$$\mathcal{A}^\bullet(M) := \{\omega \in \mathcal{A}^1(M) \mid \omega_p \in (T_p M)^\bullet \text{ for any } p \in M\}. \quad (2.6)$$

Remark 2.14. $(T_p M)^\bullet$ is the annihilator space (cf. e.g. [71], p. 102) of the radical space $T_{\circ p} M$, that is, it contains the linear forms ω_p which satisfy $\omega_p|_{T_{\circ p} M} = 0$.

Example 2.3. The radical-annihilator $T^\bullet \mathbb{R}^{r,s,t}$ of the singular semi-Euclidean manifold $\mathbb{R}^{r,s,t}$ in the Example 2.1 is:

$$T^\bullet \mathbb{R}^{r,s,t} = \bigcup_{p \in \mathbb{R}^{r,s,t}} \text{span}(\{dx^a \in T_p^* \mathbb{R}^{r,s,t} \mid a > r\}). \quad (2.7)$$

Consequently, the radical-annihilator 1-forms have the general form

$$\omega = \sum_{a=r+1}^n \omega_a dx^a, \quad (2.8)$$

and

$$\mathcal{A}^\bullet(\mathbb{R}^{r,s,t}) = \{\omega \in \mathcal{A}^1(\mathbb{R}^{r,s,t}) | \omega^i = 0, i \leq r\}. \quad (2.9)$$

2.3.3 The radical-annihilator inner product in a basis

Let us consider an inner product space (V, g) , and a basis $(e_a)_{a=1}^n$ of V in which g takes the diagonal form $g = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_a \in \mathbb{R}$ for all $1 \leq a \leq n$. The inner product satisfies:

$$g_{ab} = \langle e_a, e_b \rangle = \alpha_a \delta_{ab}. \quad (2.10)$$

We also have

$$e_a^\bullet(e_b) := \langle e_a, e_b \rangle = \alpha_a \delta_{ab},$$

and, if $(e^{*a})_{a=1}^n$ is the dual basis of $(e_a)_{a=1}^n$,

$$e_a^\bullet = \alpha_a e^{*a}. \quad (2.11)$$

Proposition 2.15. *If in a basis the inner product has the form $g_{ab} = \alpha_a \delta_{ab}$, then*

$$g_\bullet^{ab} = \frac{1}{\alpha_a} \delta^{ab}, \quad (2.12)$$

for all a so that $\alpha_a \neq 0$.

Proof. Since

$$\langle\langle e_a^\bullet, e_b^\bullet \rangle\rangle_\bullet = \langle e_a, e_b \rangle = \alpha_a \delta_{ab},$$

and in the same time

$$\langle\langle e_a^\bullet, e_b^\bullet \rangle\rangle_\bullet = \alpha_a \alpha_b \langle\langle e^{*a}, e^{*b} \rangle\rangle_\bullet = \alpha_a \alpha_b g_\bullet^{ab},$$

we have that

$$\alpha_a \alpha_b g_\bullet^{ab} = \alpha_a \delta_{ab},$$

This leads, for $\alpha_a \neq 0$, to

$$g_\bullet^{ab} = \frac{1}{\alpha_a} \delta_{ab}.$$

The case when $\alpha_a = 0$ doesn't happen, since g_\bullet is defined only on $\text{im } \flat$. □

2.3.4 Radical and radical-annihilator tensors

For inner product vector spaces we define tensors that are radical in a contravariant slot, and radical-annihilator in a covariant slot, and give their characterizations.

Definition 2.16. Let T be a tensor of type (r, s) . We call it *radical* in the k -th contravariant slot if $T \in \mathcal{T}_0^{k-1}M \otimes_M T_\circ M \otimes_M \mathcal{T}_s^{r-k}M$. We call it *radical-annihilator* in the l -th covariant slot if $T \in \mathcal{T}_{l-1}^rM \otimes_M T^\bullet M \otimes_M \mathcal{T}_{s-l}^0M$.

Proposition 2.17. A tensor $T \in \mathcal{T}_s^rM$ is radical in the k -th contravariant slot if and only if its contraction $C_{s+1}^k(T \otimes \omega)$ with any radical-annihilator linear 1-form $\omega \in \mathcal{A}^1(M)$ is zero.

Proof. For simplicity, we can work on an inner product space (V, g) and consider $k = r$ (if $k < r$, we can make use of the permutation automorphisms of the tensor space \mathcal{T}_s^rV). T can be written as a sum of linear independent terms having the form $\sum_\alpha S_\alpha \otimes v_\alpha$, with $S_\alpha \in \mathcal{T}_s^{r-1}V$ and $v_\alpha \in V$. We keep only the terms with $S_\alpha \neq 0$. The contraction of the r -th contravariant slot with any $\omega \in V^\bullet$ becomes $\sum_\alpha S_\alpha \omega(v_\alpha)$.

If T is radical in the r -th contravariant slot, for all α and any $\omega \in V^\bullet$ we have $\omega(v_\alpha) = 0$, therefore $\sum_\alpha S_\alpha \omega(v_\alpha) = 0$.

Reciprocally, if $\sum_\alpha S_\alpha \omega(v_\alpha) = 0$, it follows that for any α , $S_\alpha \omega(v_\alpha) = 0$. Then, $\omega(v_\alpha) = 0$, because $S_\alpha \neq 0$. It follows that $v_\alpha \in V_\circ$. \square

Proposition 2.18. A tensor $T \in \mathcal{T}_s^rM$ is radical-annihilator in the l -th covariant slot if and only if its l -th contraction with any radical vector field is zero.

Proof. The proof goes as in Proposition 2.17. \square

Example 2.4. The inner product g is radical-annihilator in both of its slots. This means that $g \in \mathcal{A}^\bullet(M) \odot_M \mathcal{A}^\bullet(M)$.

Proof. Follows directly from the definition of TM_\circ and of radical-annihilator tensor fields. \square

Proposition 2.19. The contraction between a radical slot and a radical-annihilator slot of a tensor is zero.

Proof. Follows from the Proposition 2.17 combined with the commutativity between tensor products and linear combinations with contraction. The proof goes similar to that of the Proposition 2.17. \square

2.4 Covariant contraction of tensor fields

We don't need an inner product to define contractions between one covariant and one contravariant indices. We can use the inner product g to contract between two contravariant indices, obtaining the *contravariant contraction operator* C^{kl} (cf. e.g. [3], p. 83). On the other hand, the contraction is not always well defined for two covariant indices. We will see that we can use g_\bullet for such contractions, but this works only for vectors or tensors which are radical-annihilator in covariant slots. Fortunately, this kind of tensors turn out to be the relevant ones in the applications to singular semi-Riemannian geometry.

2.4.1 Covariant contraction on inner product spaces

Definition 2.20. We can define uniquely the *covariant contraction* or *covariant trace* operator by the following steps.

1. We define it first on tensors $T \in V^\bullet \otimes V^\bullet$, by $C_{12}T = g_\bullet^{ab}T_{ab}$. This definition is independent on the basis, because $g_\bullet \in V^{\bullet*} \otimes V^{\bullet*}$.
2. Let $T \in \mathcal{T}_s^r V$ be a tensor with $r \geq 0$ and $s \geq 2$, which satisfies $T \in V^{\otimes r} \otimes V^{*\otimes s-2} \otimes V^\bullet \otimes V^\bullet$, that is, $T(\omega_1, \dots, \omega_r, v_1, \dots, v_s) = 0$ for any $\omega_i \in V^*$, $i = 1, \dots, r$, $v_j \in V$, $j = 1, \dots, s$ whenever $v_{s-1} \in V_\circ$ or $v_s \in V_\circ$. Then, we define the covariant contraction between the last two covariant slots by the operator

$$C_{s-1s} := 1_{\mathcal{T}_{s-2}^r V} \otimes g_\bullet : \mathcal{T}_{s-2}^r V \otimes V^\bullet \otimes V^\bullet \rightarrow \mathcal{T}_{s-2}^r V,$$

where $1_{\mathcal{T}_{s-2}^r V} : \mathcal{T}_{s-2}^r V \rightarrow \mathcal{T}_{s-2}^r V$ is the identity. In a radical basis, the contraction can be expressed by

$$(C_{s-1s}T)^{a_1 \dots a_r}_{b_1 \dots b_{s-2}} := g_\bullet^{b_{s-1}b_s} T^{a_1 \dots a_r}_{b_1 \dots b_{s-2} b_{s-1} b_s}.$$

3. Let $T \in \mathcal{T}_s^r V$ be a tensor with $r \geq 0$ and $s \geq 2$, which satisfies

$$T \in V^{\otimes r} \otimes V^{*\otimes k-1} \otimes V^\bullet \otimes V^{*\otimes l-k-1} \otimes V^\bullet \otimes V^{*\otimes s-l}, \quad (2.13)$$

$1 \leq k < l \leq s$, that is, $T(\omega_1, \dots, \omega_r, v_1, \dots, v_k, \dots, v_l, \dots, v_s) = 0$ for any $\omega_i \in V^*$, $i = 1, \dots, r$, $v_j \in V$, $j = 1, \dots, s$ whenever $v_k \in V_\circ$ or $v_l \in V_\circ$. We define the contraction

$$C_{kl} : V^{\otimes r} \otimes V^{*\otimes k-1} \otimes V^\bullet \otimes V^{*\otimes l-k-1} \otimes V^\bullet \otimes V^{*\otimes s-l} \rightarrow V^{\otimes r} \otimes V^{*\otimes s-2},$$

by $C_{kl} := C_{s-1s} \circ P_{k,s-1;l,s}$, where C_{s-1s} is the contraction defined above, and $P_{k,s-1;l,s} : T \in \mathcal{T}_s^r V \rightarrow T \in \mathcal{T}_s^r V$ is the permutation isomorphisms which moves the k -th and l -th slots in the last two positions. In a basis, the components take the form

$$(C_{kl}T)^{a_1 \dots a_r}_{b_1 \dots \widehat{b_k} \dots \widehat{b_l} \dots b_s} := g_{\bullet}^{b_k b_l} T^{a_1 \dots a_r}_{b_1 \dots b_k \dots b_l \dots b_s}. \quad (2.14)$$

We denote the contraction $C_{kl}T$ of T also by

$$C(T(\omega_1, \dots, \omega_r, v_1, \dots, \bullet, \dots, \bullet, \dots, v_s))$$

or simply

$$T(\omega_1, \dots, \omega_r, v_1, \dots, \bullet, \dots, \bullet, \dots, v_s).$$

2.4.2 Covariant contraction on singular semi-Riemannian manifolds

In §2.4.1 we have seen that we can contract in two covariant slots, so long as they are radical-annihilators. The covariant contraction uses the inner product $g_{\bullet} \in V^{\bullet*} \odot V^{\bullet*}$. In Section §2.3.4 we have extended the notion of tensors which are radical-annihilator in some slots to a singular semi-Riemannian manifold (M, g) by imposing the condition that the corresponding factors in the tensor product, at $p \in M$, are from $T^{\bullet}_p M$, which is just a subset of $T_p^* M$. This allows us easily to extend the covariant contraction (cf. e.g. [3], p. 40) in radical-annihilator slots to singular semi-Riemannian manifolds.

Definition 2.21. Let $T \in \mathcal{T}_s^r M$, $s \geq 2$, be a tensor field on M , which is radical-annihilator in the k -th and l -th covariant slots, where $1 \leq k < l \leq s$. The *covariant contraction* or *covariant trace* operator is the linear operator

$$C_{kl} : \mathcal{T}_{k-1}^r M \otimes_M \mathcal{A}^{\bullet}(M) \otimes_M \mathcal{T}_{l-k-1}^0 M \otimes_M \mathcal{A}^{\bullet}(M) \otimes_M \mathcal{T}_{s-l}^0 M \rightarrow \mathcal{T}_{s-2}^r M$$

by

$$(C_{kl}T)(p) = C_{kl}(T(p))$$

in terms of the covariant contraction defined for inner product vector spaces, as in §2.4.1. In local coordinates we have

$$(C_{kl}T)^{a_1 \dots a_r}_{b_1 \dots \widehat{b_k} \dots \widehat{b_l} \dots b_s} := g_{\bullet}^{b_k b_l} T^{a_1 \dots a_r}_{b_1 \dots b_k \dots b_l \dots b_s}. \quad (2.15)$$

We denote the contraction $C_{kl}T$ of T also by

$$C(T(\omega_1, \dots, \omega_r, X_1, \dots, \bullet, \dots, \bullet, \dots, X_s))$$

or simply

$$T(\omega_1, \dots, \omega_r, X_1, \dots, \bullet, \dots, \bullet, \dots, X_s).$$

Lemma 2.22. *If T is a tensor field $T \in \mathcal{T}_s^r M$ with $r \geq 0$ and $s \geq 1$, which is radical-annihilator in the k -th covariant slot, $1 \leq k \leq s$, then its contraction with the metric tensor gives again T :*

$$\begin{aligned} T(\omega_1, \dots, \omega_r, X_1, \dots, \bullet, \dots, X_s) \langle X_k, \bullet \rangle \\ = T(\omega_1, \dots, \omega_r, X_1, \dots, X_k, \dots, X_s) \end{aligned} \quad (2.16)$$

Proof. For simplicity, we can work on an inner product space (V, g) . Let's first consider the case when $T \in \mathcal{T}_1^0 V$, in fact, $T = \omega \in V^\bullet$. Then, equation (2.16) reduces to

$$\omega(\bullet) \langle v, \bullet \rangle = \omega(v). \quad (2.17)$$

But since $\omega \in V^\bullet$, it takes the form $\omega = u^\bullet$ for $u \in V$, and $\omega(\bullet) \langle v, \bullet \rangle = \langle \omega, v^\bullet \rangle_\bullet = \langle u, v \rangle = u^\bullet(v) = \omega(v)$.

The general case is obtained from the linearity of the tensor product in the k -th covariant slot. \square

Corollary 2.23. $\langle X, \bullet \rangle \langle Y, \bullet \rangle = \langle X, Y \rangle$.

Proof. Follows from Lemma 2.22 and from $g \in \mathcal{A}^\bullet(M) \odot_M \mathcal{A}^\bullet(M)$. \square

Example 2.5. $\langle \bullet, \bullet \rangle = \text{rank } g$.

Proof. For simplicity, we can work on an inner product space (V, g) . We recall that $g \in V^\bullet \odot V^\bullet$, $g_\bullet \in V^{\bullet*} \odot V^{\bullet*}$. When restricted to V^\bullet and $V^{\bullet*}$ they are non-degenerate and inverse to one another. Since $\dim V^\bullet = \dim \ker \flat = \text{rank } g$, we obtain $\langle \bullet, \bullet \rangle = \text{rank } g$. \square

Theorem 2.24. *Let (M, g) be a singular semi-Riemannian manifold with constant signature. Let $T \in \mathcal{T}_s^r M$, $s \geq 2$, be a tensor field which is radical-annihilator in the k -th and l -th covariant slots ($1 \leq k < l \leq n$). Let $(E_a)_{a=1}^n$ be an orthogonal basis on M , so that $E_1, \dots, E_{n-\text{rank } g} \in \mathfrak{X}_0(M)$. Then*

$$\begin{aligned} T(\omega_1, \dots, \omega_r, X_1, \dots, \bullet, \dots, \bullet, \dots, X_s) \\ = \sum_{a=n-\text{rank } g+1}^n \frac{1}{\langle E_a, E_a \rangle} T(\omega_1, \dots, \omega_r, X_1, \dots, E_a, \dots, E_a, \dots, X_s), \end{aligned} \quad (2.18)$$

for any $X_1, \dots, X_s \in \mathfrak{X}(M)$, $\omega_1, \dots, \omega_r \in \mathcal{A}^1(M)$.

Proof. For simplicity, we will work on an inner product space (V, g) . From the Proposition 2.15 we recall that g_\bullet is diagonal and $g_\bullet^{aa} = \frac{1}{g_{aa}}$, for $a > n - \text{rank } g$. Therefore

$$\begin{aligned} g_\bullet^{ab} T(\omega_1, \dots, \omega_r, v_1, \dots, E_a, \dots, E_b, \dots, v_s) \\ = \sum_{a=n-\text{rank } g+1}^n \frac{1}{\langle E_a, E_a \rangle} T(\omega_1, \dots, \omega_r, v_1, \dots, E_a, \dots, E_a, \dots, v_s). \end{aligned}$$

□

Remark 2.25. Since in fact

$$\langle\langle \omega_1, \omega_2 \rangle\rangle_\bullet = \sum_{a=n-\text{rank } g+1}^n \frac{\omega_1(E_a) \omega_2(E_a)}{\langle E_a, E_a \rangle}, \quad (2.19)$$

for any radical-annihilator 1-forms $\omega_1, \omega_2 \in \mathcal{A}^\bullet(M)$, it follows that if we define the contraction alternatively by the equation (2.18), the definition is independent on the frame $(E_a)_{a=1}^n$.

Remark 2.26. On regions of constant signature, the covariant contraction of a smooth tensor is smooth. But at the points where the signature changes, the contraction is not necessarily smooth, because the inverse of the metric becomes divergent at the points where the signature changes, as it follows from Proposition 2.15. The fact that $g_{\bullet p} \in (T^\bullet_p M)^* \odot (T^\bullet_p M)^*$ raises some problems, because the union of $(T^\bullet_p M)^*$ does not form a bundle, and for g_\bullet the notions of continuity and smoothness don't even make sense.

Counterexample 2.27. The covariant contraction of the two indices of the metric tensor at a point $p \in M$ is $g_p(\bullet, \bullet) = \text{rank } g(p)$ (see Example 2.5). When $\text{rank } g(p)$ is not constant, $g_p(\bullet, \bullet)$ is discontinuous.

On the other hand, the following example shows that it is possible to have smooth contractions even when the signature changes:

Example 2.6. If $X \in \mathfrak{X}(M)$ and $\omega \in \mathcal{A}^\bullet(M)$, $C_{12}(\omega \otimes_M X^\flat) = \langle\langle \omega, X^\flat \rangle\rangle_\bullet = \omega(X)$ and it is smooth, even if the signature is variable.

Remark 2.28. Since the points where the signature doesn't change form a dense subset of M (Remark 2.2), it makes sense to impose the condition of smoothness of the covariant contraction of a smooth tensor. To check smoothness, we simply check whether the extension by continuity of the contraction is smooth.

2.5 The Koszul form

For convenience, we name *Koszul form* the right member of the Koszul formula (see *e.g.* [3], p. 61):

Definition 2.29 (The Koszul form, see *e.g.* [19], p. 263). *The Koszul form* is defined as

$$\begin{aligned} \mathcal{K} : \mathfrak{X}(M)^3 &\rightarrow \mathbb{R}, \\ \mathcal{K}(X, Y, Z) &:= \frac{1}{2} \{ X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\ &\quad - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \}. \end{aligned} \quad (2.20)$$

The Koszul formula becomes

$$\langle \nabla_X Y, Z \rangle = \mathcal{K}(X, Y, Z), \quad (2.21)$$

and for non-degenerate metric, the unique Levi-Civita connection is obtained by raising the 1-form $\mathcal{K}(X, Y, \cdot)$:

$$\nabla_X Y = \mathcal{K}(X, Y, \cdot)^\sharp. \quad (2.22)$$

If the metric is degenerate, then this is not in general possible. We can raise $\mathcal{K}(X, Y, \cdot)$ on regions of constant signature, and what we obtain is what Kupeli ([19], p. 261–262) called *Koszul derivative* – which is in general not a connection and is not unique. Kupeli’s construction is done only for singular semi-Riemannian manifolds with metrics with constant signature, which satisfy the condition of *radical-stationarity* (Definition 2.38). But if the metric changes its signature, the Koszul derivative is discontinuous at the points where the signature changes. In this chapter we would not need to use the Koszul derivative, because for our purpose it will be enough to work with the Koszul form.

2.5.1 Basic properties of the Koszul form

Let’s recall the Lie derivative of a tensor field $T \in \mathcal{T}_2^0 M$:

Definition 2.30. (see *e.g.* [5], p. 30) Let M be a differentiable manifold. Recall that the *Lie derivative* of a tensor field $T \in \mathcal{T}_2^0 M$ with respect to a vector field $Z \in \mathfrak{X}(M)$ is given by

$$(\mathcal{L}_Z T)(X, Y) := ZT(X, Y) - T([Z, X], Y) - T(X, [Z, Y]) \quad (2.23)$$

for any $X, Y \in \mathfrak{X}(M)$.

The following properties of the Koszul form correspond directly to standard properties of the Levi-Civita connection of a non-degenerate metric (*cf. e.g.* [3], p. 61). We prove them explicitly here, because in the case of degenerate metric the proofs need to avoid using the Levi-Civita connection and the index raising. These properties will turn out to be important for what it follows.

Theorem 2.31. *The Koszul form of a singular semi-Riemannian manifold (M, g) has, for any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$, the following properties:*

1. *It is additive and \mathbb{R} -linear in each of its arguments.*

2. *It is $\mathcal{F}(M)$ -linear in the first argument:*

$$\mathcal{K}(fX, Y, Z) = f\mathcal{K}(X, Y, Z).$$

3. *Satisfies the Leibniz rule:*

$$\mathcal{K}(X, fY, Z) = f\mathcal{K}(X, Y, Z) + X(f)\langle Y, Z \rangle.$$

4. *It is $\mathcal{F}(M)$ -linear in the third argument:*

$$\mathcal{K}(X, Y, fZ) = f\mathcal{K}(X, Y, Z).$$

5. *It is metric:*

$$\mathcal{K}(X, Y, Z) + \mathcal{K}(X, Z, Y) = X\langle Y, Z \rangle.$$

6. *It is symmetric or torsionless:*

$$\mathcal{K}(X, Y, Z) - \mathcal{K}(Y, X, Z) = \langle [X, Y], Z \rangle.$$

7. *Relation with the Lie derivative of g :*

$$\mathcal{K}(X, Y, Z) + \mathcal{K}(Z, Y, X) = (\mathcal{L}_Y g)(Z, X).$$

8. $\mathcal{K}(X, Y, Z) + \mathcal{K}(Y, Z, X) = Y\langle Z, X \rangle + \langle [X, Y], Z \rangle.$

Proof. (1) Follows from Definition 2.29, and from the linearity of g , of the action of vector fields on scalars, and of the Lie brackets.

$$\begin{aligned}
(2) \quad 2\mathcal{K}(fX, Y, Z) &= fX\langle Y, Z \rangle + Y\langle Z, fX \rangle - Z\langle fX, Y \rangle \\
&\quad - \langle fX, [Y, Z] \rangle + \langle Y, [Z, fX] \rangle + \langle Z, [fX, Y] \rangle \\
&= fX\langle Y, Z \rangle + Y(f\langle Z, X \rangle) - Z(f\langle X, Y \rangle) \\
&\quad - f\langle X, [Y, Z] \rangle + \langle Y, f[Z, X] \rangle + Z(f)X \\
&\quad + \langle Z, f[X, Y] - Y(f)X \rangle \\
&= fX\langle Y, Z \rangle + fY\langle Z, X \rangle \\
&\quad + Y(f)\langle Z, X \rangle - fZ\langle X, Y \rangle \\
&\quad - Z(f)\langle X, Y \rangle - f\langle X, [Y, Z] \rangle + f\langle Y, [Z, X] \rangle \\
&\quad + Z(f)\langle Y, X \rangle + f\langle Z, [X, Y] \rangle - Y(f)\langle Z, X \rangle \\
&= fX\langle Y, Z \rangle + fY\langle Z, X \rangle - fZ\langle X, Y \rangle \\
&\quad - f\langle X, [Y, Z] \rangle + f\langle Y, [Z, X] \rangle + f\langle Z, [X, Y] \rangle \\
&= 2f\mathcal{K}(X, Y, Z)
\end{aligned}$$

$$\begin{aligned}
(3) \quad 2\mathcal{K}(X, fY, Z) &= X\langle fY, Z \rangle + fY\langle Z, X \rangle - Z\langle X, fY \rangle \\
&\quad - \langle X, [fY, Z] \rangle + \langle fY, [Z, X] \rangle + \langle Z, [X, fY] \rangle \\
&= X(f)\langle Y, Z \rangle + fX\langle Y, Z \rangle \\
&\quad + fY\langle Z, X \rangle - Z(f)\langle X, Y \rangle \\
&\quad - fZ\langle X, Y \rangle - f\langle X, [Y, Z] \rangle + Z(f)\langle X, Y \rangle \\
&\quad + f\langle Y, [Z, X] \rangle + f\langle Z, [X, Y] \rangle + X(f)\langle Z, Y \rangle \\
&= f(X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle) \\
&\quad - \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle \\
&\quad + X(f)(\langle Y, Z \rangle + \langle Z, Y \rangle) \\
&= 2(f\mathcal{K}(X, Y, Z) + X(f)\langle Y, Z \rangle)
\end{aligned}$$

$$\begin{aligned}
(4) \quad 2\mathcal{K}(X, Y, fZ) &= X\langle Y, fZ \rangle + Y\langle fZ, X \rangle - fZ\langle X, Y \rangle \\
&\quad - \langle X, [Y, fZ] \rangle + \langle Y, [fZ, X] \rangle + \langle fZ, [X, Y] \rangle \\
&= fX\langle Y, Z \rangle + X(f)\langle Y, Z \rangle \\
&\quad + fY\langle Z, X \rangle + Y(f)\langle Z, X \rangle \\
&\quad - fZ(\langle X, Y \rangle) - f\langle X, [Y, Z] \rangle - Y(f)\langle X, Z \rangle \\
&\quad + f\langle Y, [Z, X] \rangle - X(f)\langle Y, Z \rangle + f\langle Z, [X, Y] \rangle \\
&= fX\langle Y, Z \rangle + fY\langle Z, X \rangle - fZ(\langle X, Y \rangle) \\
&\quad - f\langle X, [Y, Z] \rangle + f\langle Y, [Z, X] \rangle + f\langle Z, [X, Y] \rangle \\
&= 2f\mathcal{K}(X, Y, Z)
\end{aligned}$$

$$\begin{aligned}
(5) \quad & 2[\mathcal{K}(X, Y, Z) + \mathcal{K}(X, Z, Y)] \\
&= X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\
&\quad - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \\
&\quad + X\langle Z, Y \rangle + Z\langle Y, X \rangle - Y\langle X, Z \rangle \\
&\quad - \langle X, [Z, Y] \rangle + \langle Z, [Y, X] \rangle + \langle Y, [X, Z] \rangle \\
&= X\langle Y, Z \rangle - \langle X, [Y, Z] \rangle \\
&\quad + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle + X\langle Y, Z \rangle \\
&\quad + \langle X, [Y, Z] \rangle - \langle Z, [X, Y] \rangle - \langle Y, [Z, X] \rangle \\
&= 2X\langle Y, Z \rangle
\end{aligned}$$

$$\begin{aligned}
(7) \quad & 2[\mathcal{K}(X, Y, Z) + \mathcal{K}(Z, Y, X)] \\
&= X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\
&\quad - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \\
&\quad + Z\langle Y, X \rangle + Y\langle X, Z \rangle - X\langle Z, Y \rangle \\
&\quad - \langle Z, [Y, X] \rangle + \langle Y, [X, Z] \rangle + \langle X, [Z, Y] \rangle \\
&= X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\
&\quad - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \\
&\quad + Z\langle X, Y \rangle + Y\langle Z, X \rangle - X\langle Y, Z \rangle \\
&\quad + \langle Z, [X, Y] \rangle - \langle Y, [Z, X] \rangle - \langle X, [Y, Z] \rangle \\
&= 2Y\langle Z, X \rangle - 2\langle X, [Y, Z] \rangle + 2\langle Z, [X, Y] \rangle \\
&= 2(Y\langle Z, X \rangle - \langle X, \mathcal{L}_Y Z \rangle - \langle Z, \mathcal{L}_Y X \rangle) \\
&= 2(\mathcal{L}_Y g)(Z, X)
\end{aligned}$$

$$\begin{aligned}
(6) \quad & 2[\mathcal{K}(X, Y, Z) - \mathcal{K}(Y, X, Z)] \\
&= X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\
&\quad - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \\
&\quad - Y\langle X, Z \rangle - X\langle Z, Y \rangle + Z\langle Y, X \rangle \\
&\quad + \langle Y, [X, Z] \rangle - \langle X, [Z, Y] \rangle - \langle Z, [Y, X] \rangle \\
&= X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\
&\quad - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \\
&\quad - Y\langle Z, X \rangle - X\langle Y, Z \rangle + Z\langle X, Y \rangle \\
&\quad - \langle Y, [Z, X] \rangle + \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle \\
&= 2\langle Z, [X, Y] \rangle = 2\langle [X, Y], Z \rangle
\end{aligned}$$

(8) By subtracting (6) from (5), we obtain

$$\mathcal{K}(Y, X, Z) + \mathcal{K}(X, Z, Y) = X\langle Y, Z \rangle - \langle [X, Y], Z \rangle.$$

By applying the permutation $(X, Y, Z) \mapsto (Y, X, Z)$ we get

$$\mathcal{K}(X, Y, Z) + \mathcal{K}(Y, Z, X) = Y\langle Z, X \rangle + \langle [X, Y], Z \rangle.$$

□

Remark 2.32. If $U \subseteq M$ is an open set in M and $(E_a)_{a=1}^n \subset \mathfrak{X}(U)$ are vector fields on U forming a frame of $T_p U$ at each $p \in U$, then

$$\begin{aligned} \mathcal{K}_{abc} &:= \mathcal{K}(E_a, E_b, E_c) \\ &= \frac{1}{2} \{ E_a(g_{bc}) + E_b(g_{ca}) - E_c(g_{ab}) - g_{as} \mathcal{C}_{bc}^s + g_{bs} \mathcal{C}_{ca}^s + g_{cs} \mathcal{C}_{ab}^s \}, \end{aligned} \quad (2.24)$$

where $g_{ab} = \langle E_a, E_b \rangle$ and \mathcal{C}_{ab}^c are the coefficients of the Lie bracket of vector fields (see e.g. [72], p. 107), $[E_a, E_b] = \mathcal{C}_{ab}^c E_c$.

The equations (5 – 8) in Theorem 2.31 become in the basis $(E_a)_{a=1}^n$:

$$\begin{aligned} (5') \quad & \mathcal{K}_{abc} + \mathcal{K}_{acb} = E_a(g_{bc}). \\ (7') \quad & \mathcal{K}_{abc} + \mathcal{K}_{cba} = (\mathcal{L}_{E_b} g)_{ca}. \\ (6') \quad & \mathcal{K}_{abc} - \mathcal{K}_{bac} = g_{sc} \mathcal{C}_{ab}^s. \\ (8') \quad & \mathcal{K}_{abc} + \mathcal{K}_{bca} = E_b(g_{ca}) + g_{sc} \mathcal{C}_{ab}^s. \end{aligned}$$

If $E_a = \partial_a := \frac{\partial}{\partial x^a}$ for all $a \in \{1, \dots, n\}$ are the partial derivatives in a coordinate system, $[\partial_a, \partial_b] = 0$ and the equation (2.24) reduces to

$$\mathcal{K}_{abc} = \mathcal{K}(\partial_a, \partial_b, \partial_c) = \frac{1}{2} (\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}), \quad (2.25)$$

which are Christoffel's symbols of the first kind (cf. e.g. [5], p. 40).

Corollary 2.33. *Let $X, Y \in \mathfrak{X}(M)$ two vector fields. The map $\mathcal{K}(X, Y, _) : \mathfrak{X}(M) \rightarrow \mathbb{R}$ defined as*

$$\mathcal{K}(X, Y, _)(Z) := \mathcal{K}(X, Y, Z) \quad (2.26)$$

is a differential 1-form.

Proof. It is a direct consequence of Theorem 2.31, properties (1) and (4). □

Corollary 2.34. *If $X, Y \in \mathfrak{X}(M)$ and $W \in \mathfrak{X}_o(M)$, then*

$$\mathcal{K}(X, Y, W) = \mathcal{K}(Y, X, W) = -\mathcal{K}(X, W, Y) = -\mathcal{K}(Y, W, X). \quad (2.27)$$

Proof. From Theorem 2.31, property (6),

$$\mathcal{K}(X, Y, W) = \mathcal{K}(Y, X, W) + \langle [X, Y], W \rangle = \mathcal{K}(Y, X, W). \quad (2.28)$$

From Theorem 2.31, property (5),

$$\mathcal{K}(X, Y, W) = -\mathcal{K}(X, W, Y) + X\langle Y, W \rangle = -\mathcal{K}(X, W, Y) \quad (2.29)$$

and

$$\mathcal{K}(Y, X, W) = -\mathcal{K}(Y, W, X). \quad (2.30)$$

□

2.6 The covariant derivative

2.6.1 The lower covariant derivative of vector fields

Definition 2.35 (The lower covariant derivative). The *lower covariant derivative* of a vector field Y in the direction of a vector field X is the differential 1-form $\nabla_X^b Y \in \mathcal{A}^1(M)$ defined as

$$(\nabla_X^b Y)(Z) := \mathcal{K}(X, Y, Z) \quad (2.31)$$

for any $Z \in \mathfrak{X}(M)$. The *lower covariant derivative operator* is the operator

$$\nabla^b : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathcal{A}^1(M) \quad (2.32)$$

which associates to each $X, Y \in \mathfrak{X}(M)$ the differential 1-form $\nabla_X^b Y$.

Remark 2.36. Unlike the case of the covariant derivative defined when the metric is non-degenerate, the result of applying the lower covariant derivative to a vector field is not another vector field, but a differential 1-form. When the metric is non-degenerate the two are equivalent by changing the type of the 1-form $\nabla_X^b Y$ into a vector field $\nabla_X Y = (\nabla_X^b Y)^\sharp$. Similar objects mapping vector fields to 1-forms were used in *e.g.* [34], p. 464–465. The lower covariant derivative doesn't require a non-degenerate metric, and it will be very useful in what follows.

The following properties correspond to standard properties of the Levi-Civita connection of a non-degenerate metric (*cf.* *e.g.* [3], p. 61), and are extended here to the case when the metric can be degenerate.

Theorem 2.37. *The lower covariant derivative operator ∇^b of vector fields defined on a singular semi-Riemannian manifold (M, g) has the following properties:*

1. It is additive and \mathbb{R} -linear in both of its arguments.

2. It is $\mathcal{F}(M)$ -linear in the first argument:

$$\nabla_{fX}^b Y = f \nabla_X^b Y.$$

3. Satisfies the Leibniz rule:

$$\nabla_X^b fY = f \nabla_X^b Y + X(f)Y^b.$$

or, explicitly,

$$(\nabla_X^b fY)(Z) = f(\nabla_X^b Y)(Z) + X(f)\langle Y, Z \rangle.$$

4. It is metric:

$$(\nabla_X^b Y)(Z) + (\nabla_X^b Z)(Y) = X\langle Y, Z \rangle.$$

5. It is symmetric or torsionless:

$$\nabla_X^b Y - \nabla_Y^b X = [X, Y]^b$$

or, explicitly,

$$(\nabla_X^b Y)(Z) - (\nabla_Y^b X)(Z) = \langle [X, Y], Z \rangle.$$

6. Relation with the Lie derivative of g :

$$(\nabla_X^b Y)(Z) + (\nabla_Z^b Y)(X) = (\mathcal{L}_Y g)(Z, X).$$

7. $(\nabla_X^b Y)(Z) + (\nabla_Y^b Z)(X) = Y\langle Z, X \rangle + \langle [X, Y], Z \rangle.$

for any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$.

Proof. Follows from the direct application of Theorem 2.31. □

2.6.2 Radical-stationary singular semi-Riemannian manifolds

The radical-stationary singular semi-Riemannian manifolds of constant signature were introduced by Kupeli in [19], p. 259–260, where he called them singular semi-Riemannian manifolds. Later, in [20] Definition 3.1.3, he named them “stationary singular semi-Riemannian manifolds”. Here we use the term “radical-stationary singular semi-Riemannian manifolds” to avoid possible confusion, since the word “stationary” is used in general for manifolds admitting a Killing vector field, and in particular for space-times invariant at time translation. Kupeli introduced them to ensure the existence of the Koszul derivative. Our need is different, since we don’t rely on Kupeli’s Koszul derivative.

Definition 2.38 (cf. [20] Definition 3.1.3). A singular semi-Riemannian manifold (M, g) is *radical-stationary* if it satisfies the condition

$$\mathcal{K}(X, Y, \cdot) \in \mathcal{A}^\bullet(M), \quad (2.33)$$

for any $X, Y \in \mathfrak{X}(M)$.

Remark 2.39. The condition from Definition 2.38 means that $\mathcal{K}(X, Y, W_p) = 0$ for any $X, Y \in \mathfrak{X}(M)$ and $W_p \in \mathfrak{X}_o(M_p)$, $p \in M$.

Corollary 2.40. If (M, g) is radical-stationary and $X, Y \in \mathfrak{X}(M)$ and $W \in \mathfrak{X}_o(M)$, then

$$\mathcal{K}(X, Y, W) = \mathcal{K}(Y, X, W) = -\mathcal{K}(X, W, Y) = -\mathcal{K}(Y, W, X) = 0. \quad (2.34)$$

Proof. Follows directly from the Corollary 2.34. \square

Remark 2.41. The condition (2.33) can be expressed in terms of the lower derivative as

$$\nabla_X^b Y \in \mathcal{A}^\bullet(M), \quad (2.35)$$

for any $X, Y \in \mathfrak{X}(M)$.

2.6.3 The covariant derivative of differential 1-forms

For non-degenerate metrics the covariant derivative of a differential 1-form is defined in terms of $\nabla_X Y$ (cf. e.g. [70], p. 70) by

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \quad (2.36)$$

In order to generalize this formula to the case of degenerate metrics, we need to express $\omega(\nabla_X Y)$ in terms of $\nabla_X^b Y$. We can use the identity

$$\omega(\nabla_X Y) = \langle \nabla_X Y, \omega^\sharp \rangle \quad (2.37)$$

and rewrite it in a way compatible to the degenerate case as

$$\omega(\nabla_X Y) = \langle \nabla_X Y, \bullet \rangle \langle \omega^\sharp, \bullet \rangle \quad (2.37')$$

Remark 2.42. If the metric is degenerate, we need to be allowed to define the contraction $\mathcal{K}(X, Y, \bullet)\omega(\bullet)$. This is possible on radical-stationary singular semi-Riemannian manifolds – since $\nabla_X^b Y$ is radical-annihilator – if the differential form ω is radical-annihilator too.

We can therefore give the following definition:

Definition 2.43. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the covariant derivative of a radical-annihilator 1-form $\omega \in \mathcal{A}^\bullet(M)$ in the direction of a vector field $X \in \mathfrak{X}(M)$ by

$$\nabla : \mathfrak{X}(M) \times \mathcal{A}^\bullet(M) \rightarrow A_d^1(M) \quad (2.38)$$

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - \langle \nabla_X^\flat Y, \omega \rangle_\bullet, \quad (2.39)$$

where $A_d^1(M)$ is the set of sections of T^*M smooth at the points of M where the signature is constant.

Proposition 2.44. *If (M, g) is radical-stationary and $\omega \in \mathcal{A}^\bullet(M)$ is a radical-annihilator 1-form, then for any $X \in \mathfrak{X}(M)$ and $p \in M - M_l$, $\nabla_X \omega_p \in T_p^\bullet M$.*

Proof. It follows from the Definition 2.43. Let U be a neighborhood of p where g has constant signature, and let $W \in \mathfrak{X}(U)$ so that $W_p \in T_{p^\circ} M$. Then, on U , $(\nabla_X \omega)(W) = X(\omega(W)) - \langle \nabla_X^\flat W, \omega \rangle_\bullet = 0$. \square

Corollary 2.45. *If $\nabla_X \omega$ is smooth, then it is a radical-annihilator differential 1-form, $\nabla_X \omega \in \mathcal{A}^\bullet(M)$.*

Proof. Follows from Proposition 2.44 because of continuity. \square

Definition 2.46. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the following vector spaces of differential forms having smooth covariant derivatives:

$$\mathcal{A}^{\bullet 1}(M) = \{\omega \in \mathcal{A}^\bullet(M) | (\forall X \in \mathfrak{X}(M)) \nabla_X \omega \in \mathcal{A}^\bullet(M)\}, \quad (2.40)$$

$$\mathcal{A}^{\bullet k}(M) := \bigwedge_M^k \mathcal{A}^{\bullet 1}(M). \quad (2.41)$$

The following theorem extends some properties of the covariant derivative known from the non-degenerate case (cf. e.g. [3], p. 59).

Theorem 2.47. *The covariant derivative operator ∇ of differential 1-forms defined on a radical-stationary semi-Riemannian manifold (M, g) has the following properties:*

1. *It is additive and \mathbb{R} -linear in both of its arguments.*
2. *It is $\mathcal{F}(M)$ -linear in the first argument:*

$$\nabla_{fX} \omega = f \nabla_X \omega.$$

3. It satisfies the Leibniz rule:

$$\nabla_X f\omega = f\nabla_X\omega + X(f)\omega.$$

4. It commutes with the lowering operator:

$$\nabla_X Y^\flat = \nabla_X^\flat Y.$$

for any $X, Y \in \mathfrak{X}(M)$, $\omega \in \mathcal{A}^\bullet(M)$ and $f \in \mathcal{F}(M)$.

Proof. The property (1) follows from the direct application of Theorem 2.37 to the Definition 2.43.

For property (2),

$$(\nabla_{fX}\omega)(Y) = fX(\omega(Y)) - \langle\langle \nabla_{fX}^\flat Y, \omega \rangle\rangle_\bullet = f(\nabla_X\omega)(Y). \quad (2.42)$$

Property (3) results by

$$\begin{aligned} (\nabla_X f\omega)(Y) &= X(f\omega(Y)) - \langle\langle \nabla_X^\flat Y, f\omega \rangle\rangle_\bullet \\ &= X(f)\omega(Y) + fX(\omega(Y)) - f\langle\langle \nabla_X^\flat Y, \omega \rangle\rangle_\bullet \\ &= f(\nabla_X\omega)(Y) + X(f)\omega(Y). \end{aligned} \quad (2.43)$$

For property (4), we apply Definition 2.43 to $\omega = Y^\flat$. Let $Z \in \mathfrak{X}(M)$. Then,

$$\begin{aligned} (\nabla_X Y^\flat)(Z) &= X(Y^\flat(Z)) - \langle\langle \nabla_X^\flat Z, Y^\flat \rangle\rangle_\bullet \\ &= X\langle Y, Z \rangle - (\nabla_X^\flat Z)(Y) \\ &= (\nabla_X^\flat Y)(Z), \end{aligned} \quad (2.44)$$

where the last identity follows from Theorem 4 property (4). \square

Corollary 2.48. *Let (M, g) be a radical-stationary semi-Riemannian manifold, and*

$$\mathcal{F}^\bullet(M) = \{f \in \mathcal{F}(M) | df \in \mathcal{A}^{\bullet+1}(M)\}. \quad (2.45)$$

Then, $\mathcal{A}^{\bullet+k}(M)$ from Definition 2.46 are $\mathcal{F}^\bullet(M)$ -modules of differential forms.

Proof. From Theorem 2.47 property (3) follows that for any $f \in \mathcal{F}^\bullet(M)$ and $\omega \in \mathcal{A}^{\bullet+k}(M)$, $f\omega \in \mathcal{A}^{\bullet+k}(M)$. \square

2.6.4 The covariant derivative of differential forms

We define now the covariant derivative for tensors which are covariant and radical annihilator in all their slots, in particular on differential forms (generalizing the corresponding formulas from the non-degenerate case, see *e.g.* [70], p. 70).

Definition 2.49. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the covariant derivative of tensors of type $(0, s)$ as the operator

$$\nabla : \mathfrak{X}(M) \times \otimes_M^s \mathcal{A}^{\bullet 1}(M) \rightarrow \otimes_M^s \mathcal{A}^{\bullet 1}(M) \quad (2.46)$$

acting by

$$\nabla_X(\omega_1 \otimes \dots \otimes \omega_s) := \nabla_X(\omega_1) \otimes \dots \otimes \omega_s + \dots + \omega_1 \otimes \dots \otimes \nabla_X(\omega_s) \quad (2.47)$$

In particular,

Definition 2.50. On a radical-stationary semi-Riemannian manifold (M, g) we define the covariant derivative of k -differential forms by

$$\nabla : \mathfrak{X}(M) \times \mathcal{A}^{\bullet k}(M) \rightarrow \mathcal{A}^{\bullet k}(M), \quad (2.48)$$

acting by

$$\nabla_X(\omega_1 \wedge \dots \wedge \omega_s) := \nabla_X(\omega_1) \wedge \dots \wedge \omega_s + \dots + \omega_1 \wedge \dots \wedge \nabla_X(\omega_s) \quad (2.49)$$

Theorem 2.51. The covariant derivative of a tensor $T \in \otimes_M^k \mathcal{A}^{\bullet 1}(M)$ on a radical-stationary semi-Riemannian manifold (M, g) satisfies the formula

$$\begin{aligned} (\nabla_X T)(Y_1, \dots, Y_k) &= X(T(Y_1, \dots, Y_k)) \\ &\quad - \sum_{i=1}^k \mathcal{K}(X, Y_i, \bullet) T(Y_1, \dots, \bullet, \dots, Y_k) \end{aligned} \quad (2.50)$$

Proof. Because of linearity, it is enough to prove it for the case

$$T = \omega_1 \otimes_M \dots \otimes_M \omega_k. \quad (2.51)$$

From the Definitions 2.49 and 2.43,

$$\begin{aligned}
(\nabla_X T)(Y_1, \dots, Y_k) &= \nabla_X(\omega_1 \otimes_M \dots \otimes_M \omega_k)(Y_1, \dots, Y_k) \\
&= (\nabla_X \omega_1)(Y_1) \cdot \dots \cdot \omega_k(Y_k) + \dots \\
&\quad + \omega_1(Y_1) \cdot \dots \cdot (\nabla_X \omega_k)(Y_k) \\
&= (X(\omega_1(Y_1)) - \langle \nabla_X^b Y_1, \omega_1 \rangle_\bullet) \cdot \dots \cdot \omega_k(Y_k) + \dots \\
&\quad + \omega_1(Y_1) \cdot \dots \cdot (X(\omega_k(Y_k)) - \langle \nabla_X^b Y_k, \omega_k \rangle_\bullet) \\
&= X(\omega_1(Y_1)) \cdot \dots \cdot \omega_k(Y_k) + \dots \\
&\quad + \omega_1(Y_1) \cdot \dots \cdot X(\omega_k(Y_k)) \\
&\quad - \langle \nabla_X^b Y_1, \omega_1 \rangle_\bullet \cdot \dots \cdot \omega_k(Y_k) \\
&\quad - \omega_1(Y_1) \cdot \dots \cdot \langle \nabla_X^b Y_k, \omega_k \rangle_\bullet \\
&= X(T(Y_1, \dots, Y_k)) \\
&\quad - \sum_{i=1}^k \mathcal{K}(X, Y_i, \bullet) T(Y_1, \dots, \bullet, \dots, Y_k)
\end{aligned} \tag{2.52}$$

and the desired formula follows. \square

Corollary 2.52. *Let (M, g) be a radical-stationary semi-Riemannian manifold. The covariant derivative of a k -differential form $\omega \in \mathcal{A}^{\bullet k}(M)$ takes the form*

$$\begin{aligned}
(\nabla_X \omega)(Y_1, \dots, Y_k) &:= X(\omega(Y_1, \dots, Y_k)) \\
&\quad - \sum_{i=1}^k \mathcal{K}(X, Y_i, \bullet) \omega(Y_1, \dots, \bullet, \dots, Y_k)
\end{aligned} \tag{2.53}$$

Proof. Follows from Theorem 2.51, by verifying that the antisymmetry property of ω is maintained. \square

Corollary 2.53. *On a radical-stationary semi-Riemannian manifold (M, g) , the metric g is parallel:*

$$\nabla_X g = 0. \tag{2.54}$$

Proof. Follows from Theorems 2.51 and 2.31, property (5):

$$(\nabla_X g)(Y, Z) = X\langle Y, Z \rangle - \mathcal{K}(X, Y, \bullet)g(\bullet, Z) - \mathcal{K}(X, Z, \bullet)g(Y, \bullet) = 0. \tag{2.55}$$

\square

2.6.5 Semi-regular semi-Riemannian manifolds

An important particular type of radical-stationary semi-Riemannian manifold is provided by the semi-regular semi-Riemannian manifolds, introduced below.

Definition 2.54. A *semi-regular semi-Riemannian manifold* is a singular semi-Riemannian manifold (M, g) which satisfies

$$\nabla_X^\flat Y \in \mathcal{A}^{\bullet 1}(M) \quad (2.56)$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

Remark 2.55. By Definition 2.46, this is equivalent to saying that for any $X, Y, Z \in \mathfrak{X}(M)$

$$\nabla_X \nabla_Y^\flat Z \in \mathcal{A}^\bullet(M). \quad (2.57)$$

Remark 2.56. Recall that $\mathcal{A}^{\bullet 1}(M) \subseteq \mathcal{A}^\bullet(M)$. This means that any semi-regular semi-Riemannian manifold is also radical-stationary (cf. Definition 2.38).

Proposition 2.57. *Let (M, g) be a radical-stationary semi-Riemannian manifold. The manifold (M, g) is semi-regular if and only if for any $X, Y, Z, T \in \mathfrak{X}(M)$*

$$\mathcal{K}(X, Y, \bullet) \mathcal{K}(Z, T, \bullet) \in \mathcal{F}(M). \quad (2.58)$$

Proof. From the Definition 2.43 of the covariant derivative of 1-forms we obtain that

$$\begin{aligned} (\nabla_X \nabla_Y^\flat Z)(T) &= X((\nabla_Y^\flat Z)(T)) - \langle \nabla_X^\flat T, \nabla_Y^\flat Z \rangle_\bullet \\ &= X((\nabla_Y^\flat Z)(T)) - \mathcal{K}(X, T, \bullet) \mathcal{K}(Y, Z, \bullet). \end{aligned} \quad (2.59)$$

It follows that $(\nabla_X \nabla_Y^\flat Z)(T)$ is smooth if and only if $\mathcal{K}(X, T, \bullet) \mathcal{K}(Y, Z, \bullet)$ is. \square

2.7 Curvature of semi-regular semi-Riemannian manifolds

The standard way to define the curvature invariants is to construct the Levi-Civita connection of the metric (cf. e.g. [3], p. 59), and from this the curvature operator (cf. e.g. [3], p. 74). The Ricci tensor and the scalar curvature (cf. e.g. [3], p. 87–88) follow by contraction (cf. e.g. [3], p. 83).

Unfortunately, in the case of singular semi-Riemannian manifolds the usual road is not available, because there is no intrinsic Levi-Civita connection. But, as we shall see in this section, the Riemann curvature tensor can be obtained from the lower covariant derivative and the covariant derivative of radical-annihilator differential forms. For radical-stationary manifolds the Riemann curvature tensor thus introduced is guaranteed to be smooth only on the regions of constant signature, but for semi-regular manifolds it is smooth everywhere.

In order to obtain the Ricci curvature tensor, and further the scalar curvature, we need to contract the Riemann curvature tensor in two covariant indices. Because the metric may be degenerate, this covariant contraction can be defined only if the Riemann curvature tensor is radical-annihilator in its slots. We will see that this is the case, and in §2.7.3 we define the Ricci tensor and the scalar curvature.

2.7.1 Riemann curvature of semi-regular semi-Riemannian manifolds

Definition 2.58. Let (M, g) be a radical-stationary semi-Riemannian manifold. We define the *lower Riemann curvature operator* as

$$\mathcal{R}^b : \mathfrak{X}(M)^3 \rightarrow A_d^1(M) \quad (2.60)$$

$$\mathcal{R}_{XY}^b Z := \nabla_X \nabla_Y^b Z - \nabla_Y \nabla_X^b Z - \nabla_{[X, Y]}^b Z \quad (2.61)$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Definition 2.59. We define the *Riemann curvature tensor* as

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathbb{R}, \quad (2.62)$$

$$R(X, Y, Z, T) := (\mathcal{R}_{XY}^b Z)(T) \quad (2.63)$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$.

Remark 2.60. The Riemann curvature tensor from Definition 2.59 generalizes the Riemann curvature tensor $R(X, Y, Z, T) := \langle R_{XY} Z, T \rangle$ known from semi-Riemannian geometry (cf. e.g. [3], p. 75).

Remark 2.61. It follows from the Definition 2.59 that

$$R(X, Y, Z, T) = (\nabla_X \nabla_Y^b Z)(T) - (\nabla_Y \nabla_X^b Z)(T) - (\nabla_{[X, Y]}^b Z)(T) \quad (2.64)$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$.

Theorem 2.62. Let (M, g) be a semi-regular semi-Riemannian manifold. The Riemann curvature is a smooth tensor field $R \in \mathcal{T}_4^0 M$.

Proof. Remember from Theorem 2.37, property (1) that the lower covariant derivative for vector fields is additive and \mathbb{R} -linear in both of its arguments. From the same Theorem 2.47 property (1), we recall that the covariant derivative for differential 1-forms is additive and \mathbb{R} -linear in both of its arguments. By combining the two, it follows the *additivity and \mathbb{R} -linearity* of the Riemann curvature R in all of its four arguments.

We will show now that R is $\mathcal{F}(M)$ -linear in its four arguments. The proof goes almost similar to the non-degenerate case, but we will give it explicitly, because in our proof we need to avoid any use of the Levi-Civita connection or of the inverse of the metric tensor, for example index raising.

We apply the properties of the lower covariant derivative for vector fields, as exposed in Theorem 2.47 properties (2)-(4), and those of the covariant derivative for differential 1-forms, as known from Theorem 2.47, properties (2)-(4), to verify that for any function $f \in \mathcal{F}(M)$, $R(fX, Y, Z, T) = R(X, fY, Z, T) = R(X, Y, fZ, T) = R(X, Y, Z, fT) = fR(X, Y, Z, T)$.

Since $[fX, Y] = f[X, Y] - Y(f)X$,

$$\begin{aligned}
 R(fX, Y, Z, T) &= (\nabla_{fX} \nabla_Y^b Z)(T) - (\nabla_Y \nabla_{fX}^b Z)(T) - (\nabla_{[fX, Y]}^b Z)(T) \\
 &= f(\nabla_X \nabla_Y^b Z)(T) - (\nabla_Y (f \nabla_X^b Z))(T) \\
 &\quad - (\nabla_{f[X, Y] - Y(f)X}^b Z)(T) \\
 &= f(\nabla_X \nabla_Y^b Z)(T) - f(\nabla_Y \nabla_X^b Z)(T) \\
 &\quad - Y(f)(\nabla_X^b Z)(T) - f(\nabla_{[X, Y]}^b Z)(T) \\
 &\quad + Y(f)(\nabla_X^b Z)(T) \\
 &= fR(X, Y, Z, T).
 \end{aligned}$$

The Definition 2.59 implies that $R(X, Y, Z, T) = -R(Y, X, Z, T)$, which leads immediately to

$$R(X, fY, Z, T) = fR(X, Y, Z, T). \quad (2.65)$$

$$\begin{aligned}
 R(X, Y, fZ, T) &= (\nabla_X \nabla_Y^b fZ)(T) - (\nabla_Y \nabla_X^b fZ)(T) - (\nabla_{[X, Y]}^b fZ)(T) \\
 &= (\nabla_X (f \nabla_Y^b Z + Y(f)Z))(T) \\
 &\quad - (\nabla_Y (f \nabla_X^b Z + X(f)Z))(T) \\
 &\quad - (f \nabla_{[X, Y]}^b Z + [X, Y](f)Z^b)(T) \\
 &= (\nabla_X (f \nabla_Y^b Z))(T) + (\nabla_X (Y(f)Z^b))(T) \\
 &\quad - (\nabla_Y (f \nabla_X^b Z))(T) - (\nabla_Y (X(f)Z^b))(T) \\
 &\quad - f(\nabla_{[X, Y]}^b Z)(T) - [X, Y](f)Z^b(T) \\
 &= f(\nabla_X \nabla_Y^b Z)(T) + X(f)(\nabla_Y^b Z)(T) \\
 &\quad + X(Y(f))(Z^b)(T) + Y(f)(\nabla_X Z^b)(T) \\
 &\quad - f(\nabla_Y \nabla_X^b Z)(T) - Y(f)(\nabla_X^b Z)(T) \\
 &\quad - Y(X(f))(Z^b)(T) - X(f)(\nabla_Y Z^b)(T) \\
 &\quad - f(\nabla_{[X, Y]}^b Z)(T) - [X, Y](f)Z^b(T) \\
 &= fR(X, Y, Z, T).
 \end{aligned}$$

The $\mathcal{F}(M)$ -linearity in T follows from the definition of R , observing that $\nabla_X \nabla_Y^b Z$, $\nabla_Y \nabla_X^b Z$ and $\nabla_{[X, Y]}^b Z$ are in fact differential 1-forms.

The lower covariant derivative of a smooth vector field is a smooth differential 1-form on M , therefore $\nabla_X^b Z$, $\nabla_Y^b Z$ and $\nabla_{[X,Y]}^b Z$ are smooth on M . It follows that R is also smooth on M . \square

Remark 2.63. One can write

$$\mathcal{R}^b : \mathfrak{X}(M)^2 \rightarrow \mathcal{T}_2^0 M \quad (2.66)$$

$$\mathcal{R}_{XY}^b := \nabla_X \nabla_Y^b - \nabla_Y \nabla_X^b - \nabla_{[X,Y]}^b, \quad (2.67)$$

with the amendment that

$$\mathcal{R}_{XY}^b(Z, T) := (\mathcal{R}_{XY}^b Z)(T) \quad (2.68)$$

for any $Z, T \in \mathfrak{X}(M)$.

2.7.2 The symmetries of the Riemann curvature tensor

The following proposition generalizes well-known symmetry properties of the Riemann curvature tensor of a non-degenerate metric (*cf. e.g. [3], p. 75*) to semi-regular metrics. The proofs are similar to the non-degenerate case, except that they avoid using the covariant derivative and the index raising, so we prefer to give them explicitly.

Proposition 2.64 (The symmetries of the Riemann curvature). *Let (M, g) be a semi-regular semi-Riemannian manifold. Then, for any $X, Y, Z, T \in \mathfrak{X}(M)$, the Riemann curvature has the following symmetry properties*

1. $\mathcal{R}_{XY}^b = -\mathcal{R}_{YX}^b$
2. $\mathcal{R}_{XY}^b(Z, T) = -\mathcal{R}_{XY}^b(T, Z)$
3. $\mathcal{R}_{YZ}^b X + \mathcal{R}_{ZX}^b Y + \mathcal{R}_{XY}^b Z = 0$
4. $\mathcal{R}_{XY}^b(Z, T) = \mathcal{R}_{ZT}^b(X, Y)$

Proof. (1) Follows from the Definition 2.58:

$$\begin{aligned} \mathcal{R}_{XY}^b Z &= \nabla_X \nabla_Y^b Z - \nabla_Y \nabla_X^b Z - \nabla_{[X,Y]}^b Z \\ &= -\mathcal{R}_{YX}^b Z \end{aligned}$$

(2) This is equivalent to

$$\mathcal{R}_{XY}^b(V, V) = 0 \quad (2.69)$$

for any $V \in \mathfrak{X}(M)$. From the property of the lower covariant derivative of being metric (Theorem 2.37, property (4)) it follows that

$$(\nabla_{[X,Y]}^b V)(V) = \frac{1}{2}[X, Y]\langle V, V \rangle$$

and

$$X((\nabla_Y^b V)(V)) = \frac{1}{2}XY\langle V, V \rangle.$$

From the Definition 2.43 of the covariant derivative of 1-forms we obtain that

$$(\nabla_X \nabla_Y^b V)(V) = X \left((\nabla_Y^b V)(V) \right) - \langle \nabla_X^b V, \nabla_Y^b V \rangle_\bullet. \quad (2.70)$$

By combining them we get

$$(\nabla_X \nabla_Y^b V)(V) = \frac{1}{2}XY\langle V, V \rangle - \langle \nabla_X^b V, \nabla_Y^b V \rangle_\bullet. \quad (2.71)$$

Therefore,

$$\begin{aligned} \mathcal{R}_{XY}^b(V, V) &= (\nabla_X \nabla_Y^b V)(V) - (\nabla_Y \nabla_X^b V)(V) - (\nabla_{[X,Y]}^b V)(V) \\ &= \frac{1}{2}X \left((\nabla_Y^b V)(V) \right) - \langle \nabla_X^b V, \nabla_Y^b V \rangle_\bullet \\ &\quad - \frac{1}{2}Y \left((\nabla_X^b V)(V) \right) + \langle \nabla_Y^b V, \nabla_X^b V \rangle_\bullet \\ &\quad - \frac{1}{2}[X, Y]\langle V, V \rangle = 0 \end{aligned}$$

(3) As the proof of this identity usually goes, we define the cyclic sum for any $F : \mathfrak{X}(M)^3 \rightarrow \mathcal{A}^1(M)$ by

$$\sum_{\odot} F(X, Y, Z) := F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) \quad (2.72)$$

and observe that it doesn't change at cyclic permutations of X, Y, Z . Then, from the properties of the lower covariant derivative and from Jacobi's identity,

$$\begin{aligned} \sum_{\odot} \mathcal{R}_{XY}^b Z &= \sum_{\odot} \nabla_X \nabla_Y^b Z - \sum_{\odot} \nabla_Y \nabla_X^b Z - \sum_{\odot} \nabla_{[X,Y]}^b Z \\ &= \sum_{\odot} \nabla_X \nabla_Y^b Z - \sum_{\odot} \nabla_X \nabla_Z^b Y - \sum_{\odot} \nabla_{[X,Y]}^b Z \\ &= \sum_{\odot} \nabla_X (\nabla_Y^b Z - \nabla_Z^b Y) - \sum_{\odot} \nabla_{[X,Y]}^b Z \\ &= \sum_{\odot} \nabla_X [Y, Z]^b - \sum_{\odot} \nabla_{[X,Y]}^b Z \\ &= \sum_{\odot} \nabla_X^b [Y, Z] - \sum_{\odot} \nabla_{[Y,Z]}^b X \\ &= \sum_{\odot} [X, [Y, Z]]^b = 0. \end{aligned}$$

To show (4) we apply (3) four times (as in the usual proof of the properties of the curvature):

$$\begin{aligned}\mathcal{R}_{XY}^b(Z, T) + \mathcal{R}_{YZ}^b(X, T) + \mathcal{R}_{ZX}^b(Y, T) &= 0 \\ \mathcal{R}_{YZ}^b(T, X) + \mathcal{R}_{ZT}^b(Y, X) + \mathcal{R}_{TY}^b(Z, X) &= 0 \\ \mathcal{R}_{ZT}^b(X, Y) + \mathcal{R}_{TX}^b(Z, Y) + \mathcal{R}_{XZ}^b(T, Y) &= 0 \\ \mathcal{R}_{TX}^b(Y, Z) + \mathcal{R}_{XY}^b(T, Z) + \mathcal{R}_{YT}^b(X, Z) &= 0\end{aligned}$$

then sum up, divide by 2 and get:

$$\mathcal{R}_{XY}^b(Z, T) = \mathcal{R}_{ZT}^b(X, Y).$$

□

Corollary 2.65 (see [19], p. 270). *For any $X, Y, Z \in \mathfrak{X}(M)$ and $W \in \mathfrak{X}_o(M)$, the Riemann curvature tensor R satisfies*

$$R(W, X, Y, Z) = R(X, W, Y, Z) = R(X, Y, W, Z) = R(X, Y, Z, W) = 0. \quad (2.73)$$

Proof. From the Remark 2.55, $\nabla_X \nabla_Y^b Z \in \mathcal{A}^\bullet(M)$, and from the Remark 2.41, $\nabla_X^b Y \in \mathcal{A}^\bullet(M)$, for any $X, Y, Z \in \mathfrak{X}(M)$. Therefore, $R(X, Y, Z, W) = 0$. From the symmetry properties (1) and (4) from Theorem 2.64, this property extends to all other slots of the Riemann curvature tensor. □

Corollary 2.66. *Let (M, g) be a semi-regular semi-Riemannian manifold. Then, for any $X, Y \in \mathfrak{X}(M)$, $\mathcal{R}_{XY}^b \in \mathcal{A}^{\bullet 2}(M)$ (\mathcal{R}_{XY}^b is a radical-annihilator).*

Proof. Follows from the Corollary 2.65. □

2.7.3 Ricci curvature tensor and scalar curvature

In non-degenerate semi-Riemannian geometry, the Ricci tensor is obtained by tracing the Riemann curvature, and the scalar curvature by tracing the Ricci tensor (*cf. e.g. [3], p. 87–88*). In the degenerate case, an invariant contraction can be performed only on radical-annihilator slots. Fortunately, this is the case of the Riemann tensor even in the case when the metric is degenerate (Corollary 2.65), so it is possible to define the Ricci tensor as:

Definition 2.67. Let (M, g) be a radical-stationary singular semi-Riemannian manifold with constant signature. The *Ricci curvature tensor* is defined as the covariant contraction of the Riemann curvature tensor

$$\text{Ric}(X, Y) := R(X, \bullet, Y, \bullet) \quad (2.74)$$

for any $X, Y \in \mathfrak{X}(M)$.

The symmetry of the Ricci tensor works just like in the non-degenerate case (cf. e.g. [3], p. 87):

Proposition 2.68. *The Ricci curvature tensor on a radical-stationary singular semi-Riemannian manifold with constant signature is symmetric:*

$$\text{Ric}(X, Y) = \text{Ric}(Y, X) \quad (2.75)$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. The Proposition 2.64 states that for any $X, Y, Z, T \in \mathfrak{X}(M)$, $R(X, Y, Z, T) = R(Z, T, X, Y)$. Therefore, $\text{Ric}(X, Y) = \text{Ric}(Y, X)$. \square

The scalar curvature is obtained from the Ricci tensor like in the non-degenerate case (cf. e.g. [3], p. 88):

Definition 2.69. Let (M, g) be a radical-stationary singular semi-Riemannian manifold with constant signature. The *scalar curvature* is defined as the covariant contraction of the Ricci curvature tensor

$$s := \text{Ric}(\bullet, \bullet). \quad (2.76)$$

Remark 2.70. The Ricci and the scalar curvatures are smooth for the case of radical-stationary singular semi-Riemannian manifolds having the metric with constant signature. For semi-regular semi-Riemannian manifolds, the Ricci and scalar curvatures are smooth in the regions of constant curvature, and become in general divergent as we approach the points where the signature changes.

2.8 Curvature of semi-regular semi-Riemannian manifolds II

This section contains some complements on the Riemann curvature tensor of semi-regular semi-Riemannian manifolds. A useful formula of this curvature in terms of the Koszul form is provided in §2.8.1.

In the subsection §2.8.2 we recall some results from [19] concerning the (non-unique) Koszul derivative ∇ and the associated curvature function R_∇ , and show that the curvature $\langle R_\nabla(-, -), - \rangle$ coincides with that of the Riemann curvature tensor given in §2.7.

2.8.1 Riemann curvature in terms of the Koszul form

Proposition 2.71. *For any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$ on a semi-regular semi-Riemannian manifold (M, g) :*

$$\begin{aligned} R(X, Y, Z, T) &= X((\nabla_Y^b Z)(T)) - Y((\nabla_X^b Z)(T)) - (\nabla_{[X, Y]}^b Z)(T) \\ &\quad + \langle \nabla_X^b Z, \nabla_Y^b T \rangle_\bullet - \langle \nabla_Y^b Z, \nabla_X^b T \rangle_\bullet. \end{aligned} \quad (2.77)$$

and, alternatively,

$$\begin{aligned} R(X, Y, Z, T) &= X\mathcal{K}(Y, Z, T) - Y\mathcal{K}(X, Z, T) - \mathcal{K}([X, Y], Z, T) \\ &\quad + \mathcal{K}(X, Z, \bullet)\mathcal{K}(Y, T, \bullet) - \mathcal{K}(Y, Z, \bullet)\mathcal{K}(X, T, \bullet) \end{aligned} \quad (2.78)$$

Proof. From the Definition 2.43 of the covariant derivative of 1-forms we obtain that

$$(\nabla_X \nabla_Y^b Z)(T) = X((\nabla_Y^b Z)(T)) - \langle \nabla_X^b T, \nabla_Y^b Z \rangle_\bullet, \quad (2.79)$$

therefore

$$\begin{aligned} R(X, Y, Z, T) &= (\nabla_X \nabla_Y^b Z)(T) - (\nabla_Y \nabla_X^b Z)(T) - (\nabla_{[X, Y]}^b Z)(T) \\ &= X((\nabla_Y^b Z)(T)) - Y((\nabla_X^b Z)(T)) - (\nabla_{[X, Y]}^b Z)(T) \\ &\quad + \langle \nabla_X^b Z, \nabla_Y^b T \rangle_\bullet - \langle \nabla_Y^b Z, \nabla_X^b T \rangle_\bullet. \end{aligned} \quad (2.80)$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$. The second formula (2.78) follows from the definition of the lower derivative of vector fields. \square

Remark 2.72. In a coordinate basis, the components of the Riemann curvature tensor are given by

$$R_{abcd} = \partial_a \mathcal{K}_{bcd} - \partial_b \mathcal{K}_{acd} + g_{\bullet}^{st} (\mathcal{K}_{acs} \mathcal{K}_{bdt} - \mathcal{K}_{bcs} \mathcal{K}_{adt}). \quad (2.81)$$

Proof.

$$\begin{aligned} R_{abcd} &:= R(\partial_a, \partial_b, \partial_c, \partial_d) \\ &= \partial_a \mathcal{K}(\partial_b, \partial_c, \partial_d) - \partial_b \mathcal{K}(\partial_a, \partial_c, \partial_d) - \mathcal{K}([\partial_a, \partial_b], \partial_c, \partial_d) \\ &\quad + \mathcal{K}(\partial_a, \partial_c, \bullet) \mathcal{K}(\partial_b, \partial_d, \bullet) - \mathcal{K}(\partial_b, \partial_c, \bullet) \mathcal{K}(\partial_a, \partial_d, \bullet) \\ &= \partial_a \mathcal{K}_{bcd} - \partial_b \mathcal{K}_{acd} + g_{\bullet}^{st} (\mathcal{K}_{acs} \mathcal{K}_{bdt} - \mathcal{K}_{bcs} \mathcal{K}_{adt}) \end{aligned} \quad (2.82)$$

\square

2.8.2 Relation with Kupeli's curvature function

Through the work of Demir Kupeli [19] we have seen that for a radical-stationary singular semi-Riemannian manifold (with constant signature) (M, g) there is always a Koszul derivative ∇ , from whose curvature function R_∇ we can construct a tensor field $\langle R_\nabla(-, -), -, - \rangle$. We may wonder how is $\langle R_\nabla(-, -), -, - \rangle$ related to the Riemann curvature tensor from the Definition 2.59. We will see that they coincide for a radical-stationary singular semi-Riemannian manifold.

Definition 2.73 (Koszul derivative, cf. Kupeli [19], p. 261). A *Koszul derivative* on a radical-stationary semi-Riemannian manifold with constant signature is an operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ which satisfies the *Koszul formula*

$$\langle \nabla_X Y, Z \rangle = \mathcal{K}(X, Y, Z). \quad (2.83)$$

Remark 2.74 (cf. Kupeli [19], p. 262). The Koszul derivative corresponds, for the non-degenerate case, to the Levi-Civita connection.

Definition 2.75 (Curvature function, cf. Kupeli [19], p. 266). The *curvature function* $R_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ of a Koszul derivative ∇ on a singular semi-Riemannian manifold with constant signature (M, g) is defined by

$$R_\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (2.84)$$

Remark 2.76. In [19], p. 266-268, it is shown that $\langle R_\nabla(-, -), -, - \rangle \in \mathcal{T}_4^0 M$ and it has the same symmetry properties as the Riemann curvature tensor of a Levi-Civita connection.

Theorem 2.77. *Let (M, g) be a radical-stationary singular semi-Riemannian manifold with constant signature, and ∇ a Koszul derivative on M . The Riemann curvature tensor is related to the curvature function by*

$$\langle R_\nabla(X, Y)Z, T \rangle = R(X, Y, Z, T) \quad (2.85)$$

for any $X, Y, Z, T \in \mathfrak{X}(M)$.

Proof. From Theorem 2.31 and Definition 2.75, applying the property of contraction with the metric from Lemma 2.22 and the Koszul formula for the Riemann curvature

tensor (2.78), we obtain

$$\begin{aligned}
\langle R_{\nabla}(X, Y)Z, T \rangle &= \langle \nabla_X \nabla_Y Z, T \rangle - \langle \nabla_Y \nabla_X Z, T \rangle - \langle \nabla_{[X, Y]} Z, T \rangle \\
&= X \langle \nabla_Y Z, T \rangle - \langle \nabla_Y Z, \nabla_X T \rangle \\
&\quad - Y \langle \nabla_X Z, T \rangle + \langle \nabla_X Z, \nabla_Y T \rangle - \langle \nabla_{[X, Y]} Z, T \rangle \\
&= X \mathcal{K}(Y, Z, T) - \mathcal{K}(Y, Z, \bullet) \mathcal{K}(X, T, \bullet) \\
&\quad - Y \mathcal{K}(X, Z, T) + \mathcal{K}(X, Z, \bullet) \mathcal{K}(Y, T, \bullet) \\
&\quad - \mathcal{K}([X, Y], Z, T) \\
&= R(X, Y, Z, T)
\end{aligned}$$

□

2.9 Examples of semi-regular semi-Riemannian manifolds

2.9.1 Diagonal metric

Let (M, g) be a singular semi-Riemannian manifold with variable signature having the property that for each point $p \in M$ there is a local coordinate system around p in which the metric takes a diagonal form $g = \text{diag}(g_{11}, \dots, g_{nn})$. According to equation (2.25), $2\mathcal{K}_{abc} = \partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}$, but since g is diagonal, we have only the following possibilities: $\mathcal{K}_{baa} = \mathcal{K}_{aba} = -\mathcal{K}_{aab} = \frac{1}{2}\partial_b g_{aa}$, for $a \neq b$, and $\mathcal{K}_{aaa} = \frac{1}{2}\partial_a g_{aa}$.

The manifold (M, g) is radical-stationary if and only if whenever $g_{aa} = 0$, $\partial_b g_{aa} = \partial_a g_{bb} = 0$.

According to Proposition 2.57, the manifold (M, g) is semi-regular if and only if

$$\sum_{\substack{s \in \{1, \dots, n\} \\ g_{ss} \neq 0}} \frac{\partial_a g_{ss} \partial_b g_{ss}}{g_{ss}}, \quad \sum_{\substack{s \in \{1, \dots, n\} \\ g_{ss} \neq 0}} \frac{\partial_s g_{aa} \partial_s g_{bb}}{g_{ss}}, \quad \sum_{\substack{s \in \{1, \dots, n\} \\ g_{ss} \neq 0}} \frac{\partial_a g_{ss} \partial_s g_{bb}}{g_{ss}} \quad (2.86)$$

are all smooth.

One way to ensure this is for instance if the functions $u, v : M \rightarrow \mathbb{R}$ defined as

$$u(p) := \begin{cases} \frac{\partial_b g_{aa}}{\sqrt{|g_{aa}|}} & g_{aa} \neq 0 \\ 0 & g_{aa} = 0 \end{cases} \quad \text{and} \quad v(p) := \begin{cases} \frac{\partial_a g_{bb}}{\sqrt{|g_{aa}|}} & g_{aa} \neq 0 \\ 0 & g_{aa} = 0 \end{cases} \quad (2.87)$$

and $\sqrt{|g_{aa}|}$ are smooth for all $a, b \in \{1, \dots, n\}$. In this case it is easy to see that all the terms of the sums in equation (2.86) are smooth.

Assume that $g = \sum_a \varepsilon_a \alpha_a^2 dx^a \otimes dx^a$, $\varepsilon_a \in \{-1, 1\}$. Then the metric is semi-regular if there is a smooth function $f_{abc} \in \mathcal{F}(M)$ with $\text{supp}(f_{abc}) \subseteq \text{supp}(\alpha_c)$ for any $a, b \in \{1, \dots, n\}$ and $c \in \{a, b\}$, and

$$\partial_a \alpha_b^2 = f_{abc} \alpha_c. \quad (2.88)$$

If $c = b$, $\partial_a \alpha_b^2 = 2\alpha_b \partial_a \alpha_b$ implies that the function is $f_{abb} = 2\partial_a \alpha_b$. In addition, this has to satisfy the condition $\partial_a \alpha_b = 0$ whenever $\alpha_b = 0$. We require the condition $\text{supp}(f_{abc}) \subseteq \text{supp}(\alpha_c)$ because for being semi-regular, a manifold has to be radical-stationary.

2.9.2 Conformally-non-degenerate metrics

Another example of semi-regular metric is given by those that can be obtained by a conformal transformation (cf. e.g. [5], p. 42) from non-degenerate metrics.

Definition 2.78. A singular semi-Riemannian manifold (M, g) is said to be *conformally non-degenerate* if there is a non-degenerate semi-Riemannian metric \tilde{g} on M and a smooth function $\Omega \in \mathcal{F}(M)$, $\Omega \geq 0$, so that $g(X, Y) = \Omega^2 \tilde{g}(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$. The manifold (M, g) is alternatively denoted by (M, \tilde{g}, Ω) .

The following proposition shows what happens to the Koszul form at a conformal transformation of the metric, similar to the non-degenerate case (cf. e.g. [5], p. 42).

Proposition 2.79. *Let (M, \tilde{g}, Ω) be a conformally non-degenerate singular semi-Riemannian manifold. Then, the Koszul form \mathcal{K} of g is related to the Koszul form $\tilde{\mathcal{K}}$ of \tilde{g} by:*

$$\mathcal{K}(X, Y, Z) = \Omega^2 \tilde{\mathcal{K}}(X, Y, Z) + \Omega [\tilde{g}(Y, Z)X + \tilde{g}(X, Z)Y - \tilde{g}(X, Y)Z] (\Omega) \quad (2.89)$$

Proof. From the Koszul formula we obtain

$$\begin{aligned} \mathcal{K}(X, Y, Z) &= \frac{1}{2} \{ X(\Omega^2 \tilde{g}(Y, Z)) + Y(\Omega^2 \tilde{g}(Z, X)) - Z(\Omega^2 \tilde{g}(X, Y)) \\ &\quad - \Omega^2 \tilde{g}(X, [Y, Z]) + \Omega^2 \tilde{g}(Y, [Z, X]) + \Omega^2 \tilde{g}(Z, [X, Y]) \} \\ &= \frac{1}{2} \{ \Omega^2 X(\tilde{g}(Y, Z)) + \tilde{g}(Y, Z)X(\Omega^2) + \Omega^2 Y(\tilde{g}(X, Z)) \\ &\quad + \tilde{g}(X, Z)Y(\Omega^2) - \Omega^2 Z(\tilde{g}(X, Y)) - \tilde{g}(X, Y)Z(\Omega^2) \\ &\quad - \Omega^2 \tilde{g}(X, [Y, Z]) + \Omega^2 \tilde{g}(Y, [Z, X]) + \Omega^2 \tilde{g}(Z, [X, Y]) \} \\ &= \Omega^2 \tilde{\mathcal{K}}(X, Y, Z) + \frac{1}{2} \{ \tilde{g}(Y, Z)X(\Omega^2) \\ &\quad + \tilde{g}(X, Z)Y(\Omega^2) - \tilde{g}(X, Y)Z(\Omega^2) \} \\ &= \Omega^2 \tilde{\mathcal{K}}(X, Y, Z) + \Omega [\tilde{g}(Y, Z)X \\ &\quad + \tilde{g}(X, Z)Y - \tilde{g}(X, Y)Z] (\Omega) \end{aligned}$$

□

Theorem 2.80. *Let (M, \tilde{g}, Ω) be a singular semi-Riemannian manifold which is conformally non-degenerate. Then, $(M, g = \Omega^2 \tilde{g})$ is a semi-regular semi-Riemannian manifold.*

Proof. The metric g is either non-degenerate, or it is 0. Therefore, the manifold (M, g) is radical-stationary.

Let $(E_a)_{a=1}^n$ be a local frame of vector fields on an open $U \subseteq M$, which is orthonormal with respect to the non-degenerate metric \tilde{g} . Then, the metric g is diagonal in $(E_a)_{a=1}^n$.

Proposition 2.79 implies that the Koszul form has the form $\mathcal{K}(X, Y, Z) = \Omega h(X, Y, Z)$, where

$$h(X, Y, Z) = \Omega \tilde{\mathcal{K}}(X, Y, Z) + [\tilde{g}(Y, Z)X + \tilde{g}(X, Z)Y - \tilde{g}(X, Y)Z](\Omega) \quad (2.90)$$

is a smooth function depending on X, Y, Z . Moreover, if $\Omega = 0$, then $h(X, Y, Z) = 0$ as well, because the first term is multiple of Ω , and the second is a partial derivative of Ω , which reaches its minimum at 0.

Theorem 2.24 says that, on the regions of constant signature, if $r = n - \text{rank } g + 1$, for any $X, Y, Z, T \in U$ and for any $a \in \{1, \dots, n\}$,

$$\begin{aligned} \mathcal{K}(X, Y, \bullet) \mathcal{K}(Z, T, \bullet) &= \sum_{a=r}^n \frac{\mathcal{K}(X, Y, E_a) \mathcal{K}(Z, T, E_a)}{g(E_a, E_a)} \\ &= \sum_{a=r}^n \frac{\Omega^2 h(X, Y, E_a) h(Z, T, E_a)}{\Omega^2 \tilde{g}(E_a, E_a)} \\ &= \sum_{a=1}^n \frac{h(X, Y, E_a) h(Z, T, E_a)}{\tilde{g}(E_a, E_a)}. \end{aligned} \quad (2.91)$$

If $\Omega = 0$, then $h(X, Y, Z) = 0$, therefore the last member does not depend on r . It follows that $\mathcal{K}(X, Y, \bullet) \mathcal{K}(Z, T, \bullet) \in \mathcal{F}(M)$, and according to Proposition 2.57, (M, g) is semi-regular. □

Chapter 3

Degenerate warped products

The text in this chapter is part of author's paper [73].

This chapter studies the degenerate warped products of singular semi-Riemannian manifolds. One main result is that a degenerate warped product of semi-regular semi-Riemannian manifolds with the warping function satisfying a certain condition is a semi-regular semi-Riemannian manifold. The main invariants of the warped product are expressed in terms of those of the factor manifolds. Examples of singular semi-Riemannian manifolds which are semi-regular are constructed as warped products. Degenerate warped products are used to define spherical warped products. As applications, cosmological models and black holes solutions with semi-regular singularities are constructed. Such singularities are compatible with the densitized version of Einstein's equation, and don't block the time evolution. In following chapters we will apply the technique developed here to resolve the singularities of the Friedmann-Lemaître-Robertson-Walker, Schwarzschild, Reissner-Nordström and Kerr-Newman spacetimes.

3.1 Introduction

The warped product provides a way to construct new semi-Riemannian manifolds from known ones [3, 74, 75]. This construction has useful applications in General Relativity, in the study cosmological models and black holes. In such models, singularities are usually present, and at such points the warping function becomes 0. The metric of the product manifold in this case becomes degenerate, and we need to apply the tools of singular semi-Riemannian geometry.

This chapter continues the study of singular semi-Riemannian manifolds [19, 20], [67], extending it to warped products. We start with a brief recall of notions related to

product manifolds in §3.2.1. Then, in §3.3 we define the degenerate warped products of singular semi-Riemannian manifolds, and study the Koszul form of the warped product in terms of the Koszul form of the factors. The main results known from the literature about the non-degenerate warped products of semi-Riemannian manifolds are recalled in §3.4. Then, in §3.5 we show that the warped products of radical-stationary manifolds are also radical-stationary, if the warping function satisfies a certain condition. Then we prove a similar result for semi-regular manifolds, which ensures the smoothness of the Riemann curvature tensor. In §3.6 we express the Riemann curvature of semi-regular warped products in terms of the factor manifolds. Then, in §3.7, we introduce the polar and spherical warped products, which allows us to construct singular semi-Riemannian manifolds with radial or spherical symmetry. We conclude in §3.8 by giving some examples of semi-regular warped products, and some applications to General Relativity. Cosmological models having the Big Bang singularity semi-regular, are proposed. Spherical solutions with semi-regular singularities are constructed in a general way. Semi-regular singularities are compatible with a densitized version of Einstein's equation, and they don't block the time evolution.

3.2 Preliminaries

3.2.1 Product manifolds

We first recall some elementary notions about the *product manifold* $B \times F$ of two differentiable manifolds B and F (cf. e.g. [3], p. 24–25).

At each point $p = (p_1, p_2)$ of the manifold $M_1 \times M_2$ the tangent space decomposes as

$$T_{(p_1, p_2)}(M_1 \times M_2) \cong T_{(p_1, p_2)}(M_1) \oplus T_{(p_1, p_2)}(M_2), \quad (3.1)$$

where $T_{(p_1, p_2)}(M_1) := T_{(p_1, p_2)}(M_1 \times p_2)$ and $T_{(p_1, p_2)}(M_2) := T_{(p_1, p_2)}(p_1 \times M_2)$.

Let $\pi_i : M_1 \times M_2 \rightarrow M_i$, for $i \in \{1, 2\}$, be the canonical projections. The *lift of the scalar field* $f_i \in \mathcal{F}(M_i)$ is the scalar field $\tilde{f}_i := f_i \circ \pi_i \in \mathfrak{X}(M_1 \times M_2)$. The *lift of the vector field* $X_i \in \mathfrak{X}(M_i)$ is the unique vector field \tilde{X}_i on $M_1 \times M_2$ satisfying $d\pi_i(\tilde{X}_i) = X_i$. We denote the set of all vector fields $X \in \mathfrak{X}(M_1 \times M_2)$ which are lifts of vector fields $X_i \in \mathfrak{X}(M_i)$ by $\mathfrak{L}(M, M_i)$. The *lift of a covariant tensor* $T \in \mathcal{T}_s^0 M_i$ is given by $\tilde{T} \in \mathcal{T}_s^0(M_1 \times M_2)$, $\tilde{T} := \pi_i^*(T)$. The *lift of a tensor* $T \in \mathcal{T}_s^1 M_i$ is given, for any $X_1, \dots, X_s \in \mathfrak{X}(M_1 \times M_2)$, by $\tilde{T} \in \mathcal{T}_s^1(M_1 \times M_2)$, $\tilde{T}(X_1, \dots, X_s) = \tilde{X}$, where $\tilde{X} \in \mathfrak{X}(M_1 \times M_2)$ is the lifting of the vector field $X \in \mathfrak{X}(M_i)$, $X = T(\pi_i(X_1), \dots, \pi_i(X_s))$.

3.3 Degenerate warped products of singular semi-Riemannian manifolds

The warped product is defined in general between two (non-degenerate) semi-Riemannian manifolds, (cf. [74], [75], [3], p. 204–211. It is straightforward to extend the definition to singular semi-Riemannian manifolds, as it is done in this section.

Definition 3.1 (generalizing [3], p. 204). Let (B, g_B) and (F, g_F) be two singular semi-Riemannian manifolds, and $f \in \mathcal{F}(B)$ a smooth function. The *warped product* of B and F with *warping function* f is the semi-Riemannian manifold

$$B \times_f F := (B \times F, \pi_B^*(g_B) + (f \circ \pi_B)\pi_F^*(g_F)), \quad (3.2)$$

where $\pi_B : B \times F \rightarrow B$ and $\pi_F : B \times F \rightarrow F$ are the canonical projections. It is customary to call B the *base* and F the *fiber* of the warped product $B \times_f F$.

We will use for all vector fields $X_B, Y_B \in \mathfrak{X}(B)$ and $X_F, Y_F \in \mathfrak{X}(F)$ the notation $\langle X_B, Y_B \rangle_B := g_B(X_B, Y_B)$ and $\langle X_F, Y_F \rangle_F := g_F(X_F, Y_F)$. The inner product on $B \times_f F$ takes, for any point $p \in B \times F$ and for any pair of tangent vectors $x, y \in T_p(B \times F)$, the explicit form

$$\langle x, y \rangle = \langle d\pi_B(x), d\pi_B(y) \rangle_B + f^2(p) \langle d\pi_F(x), d\pi_F(y) \rangle_F. \quad (3.3)$$

Remark 3.2. The degenerate warped product metric from Definition 3.1 has the form

$$ds_{B \times F}^2 = ds_B^2 + f^2 ds_F^2 \quad (3.4)$$

Remark 3.3. Definition 3.1 is a generalization of the warped product definition, which is usually given for the case when both g_B and g_F are non-degenerate and $f > 0$ (see [74], [75] and [3]). In our definition these restrictions are dropped.

Remark 3.4 (similar to [3], p. 204–205). For any $p_B \in B$, $\pi_B^{-1}(p_B) = p_B \times F$ is named the *fiber* through p_B and it is a semi-Riemannian manifold. $\pi_F|_{p_B \times F}$ is a (possibly degenerate) homothety onto F . For each $p_F \in F$, $\pi_F^{-1}(p_F) = B \times p_F$ is a semi-Riemannian manifold named the *leave* through p_F . $\pi_B|_{B \times p_F}$ is an isometry onto B . For each $(p_B, p_F) \in B \times F$, $B \times p_F$ and $p_B \times F$ are orthogonal at (p_B, p_F) . For simplicity, if a vector field is a lift, we will use sometimes the same notation if they can be distinguished from the context. For example, we will be using $\langle V, W \rangle_F := \langle \pi_F(V), \pi_F(W) \rangle_F$ for $V, W \in \mathfrak{L}(B \times F, F)$.

The following proposition recalls some evident facts used repeatedly in the proofs of the properties of warped products in [3], p. 24–25, 206.

Proposition 3.5. *Let $B \times_f F$ be a warped product, and let be the vector fields $X, Y, Z \in \mathfrak{L}(B \times F, B)$ and $U, V, W \in \mathfrak{L}(B \times F, F)$. Then*

1. $\langle X, V \rangle = 0$.
2. $[X, V] = 0$.
3. $V\langle X, Y \rangle = 0$.
4. $X\langle V, W \rangle = 2f\langle V, W \rangle_F X(f)$.

Proof. (1) and (2) are evident because the manifold is $B \times F$.

(3) $\langle X, Y \rangle = \langle X, Y \rangle_B$ is constant on fibers, and $V\langle X, Y \rangle = 0$ because V is vertical.

(4) $X\langle V, W \rangle = X(f^2\langle V, W \rangle_F) = 2f\langle V, W \rangle_F X(f)$. □

The following proposition generalizes the properties of the Levi-Civita connection for the warped product of (non-degenerate) semi-Riemannian manifolds (*cf. e.g. [3], p. 206*), to the degenerate case. We preferred to express them in terms of the Koszul form, and to give the proof explicitly, because for degenerate metric the Levi-Civita connection is not defined, and we need to avoid the index raising.

Proposition 3.6. *Let $B \times_f F$ be a warped product, and let be the vector fields $X, Y, Z \in \mathfrak{L}(B \times F, B)$ and $U, V, W \in \mathfrak{L}(B \times F, F)$. Let \mathcal{K} be the Koszul form on $B \times_f F$, and $\mathcal{K}_B, \mathcal{K}_F$ the lifts of the Koszul forms on B , respectively F . Then*

1. $\mathcal{K}(X, Y, Z) = \mathcal{K}_B(X, Y, Z)$.
2. $\mathcal{K}(X, Y, W) = \mathcal{K}(X, W, Y) = \mathcal{K}(W, X, Y) = 0$.
3. $\mathcal{K}(X, V, W) = \mathcal{K}(V, X, W) = -\mathcal{K}(V, W, X) = f\langle V, W \rangle_F X(f)$.
4. $\mathcal{K}(U, V, W) = f^2\mathcal{K}_F(U, V, W)$.

Proof. (1) and (4) follow from properties of the lifts of vector fields, the Definition 2.29 of the Koszul form, and the equation (3.3).

(2) By Definition 2.29,

$$\begin{aligned} \mathcal{K}(X, Y, W) &= \frac{1}{2} \{ X\langle Y, W \rangle + Y\langle W, X \rangle - W\langle X, Y \rangle \\ &\quad - \langle X, [Y, W] \rangle + \langle Y, [W, X] \rangle + \langle W, [X, Y] \rangle \} \end{aligned}$$

We apply the Proposition 3.5. From the relation (1),

$$\langle Y, W \rangle = \langle W, X \rangle = \langle W, [X, Y] \rangle = 0,$$

from the relation (2) $[Y, W] = [W, X] = 0$, from the relation (3) $W\langle X, Y \rangle = 0$. Therefore $\mathcal{K}(X, Y, W) = 0$.

From (5) of the Theorem 2.31 we obtain that

$$\mathcal{K}(X, W, Y) = X\langle W, Y \rangle - \mathcal{K}(X, Y, W) = 0.$$

From (6) of the Theorem 2.31 and from Proposition 3.5(2) we obtain that

$$\mathcal{K}(W, X, Y) = \mathcal{K}(X, W, Y) - \langle [X, W], Y \rangle = 0.$$

$$\begin{aligned} \mathcal{K}(X, V, W) &:= \frac{1}{2} \{ X\langle V, W \rangle + V\langle W, X \rangle - W\langle X, V \rangle \\ &\quad - \langle X, [V, W] \rangle + \langle V, [W, X] \rangle + \langle W, [X, V] \rangle \} \\ &= \frac{1}{2} X\langle V, W \rangle \end{aligned} \quad (3)$$

from Proposition 3.5, using it as in the property (2) of the present Proposition. By applying the property (4) we have $\mathcal{K}(X, V, W) = f\langle V, W \rangle_F X(f)$. From Theorem 2.31 property (6),

$$\mathcal{K}(V, X, W) = \mathcal{K}(X, V, W) - \langle [X, V], W \rangle,$$

but since $[X, V] = 0$, $\mathcal{K}(V, X, W) = f\langle V, W \rangle_F X(f)$ as well.

From Theorem 2.31 property (5),

$$\mathcal{K}(V, W, X) = V\langle W, X \rangle - \mathcal{K}(V, X, W),$$

but since $\langle W, X \rangle = 0$, the property (3) of the present Proposition shows that

$$\mathcal{K}(V, W, X) = -f\langle V, W \rangle_F X(f).$$

□

Further, we will study some properties of the warped products, in situations when the warping function f is allowed to cancel or to become negative, and when (B, g_B) and (F, g_F) are allowed to be singular and with variable signature. But first, we need to recall what we know about non-degenerate warped products of non-singular semi-Riemannian manifolds.

3.4 Non-degenerate warped products

Here we recall for comparison and without proofs some fundamental properties of non-degenerate warped products between non-singular semi-Riemannian manifolds. The main reference is [3], p. 204–211. Here, (B, g_B) and (F, g_F) are semi-Riemannian manifolds, $f \in \mathcal{F}(B)$ a smooth function so that $f > 0$, and $B \times_f F$ the warped product of B and F .

For the proofs of the next propositions, see for example [3].

Proposition 3.7 (cf. [3], p. 206–207). *Let $B \times_f F$ be a warped product, and let be the vector fields $X, Y \in \mathfrak{L}(B \times F, B)$ and $V, W \in \mathfrak{L}(B \times F, F)$. Let $\nabla, \nabla^B, \nabla^F$ be the Levi-Civita connections on $B \times_f F$, B , respectively F . Then*

1. $\nabla_X Y$ is the lift of $\nabla_X^B Y$.
2. $\nabla_X V = \nabla_V X = \frac{Xf}{f} V$.
3. $\nabla_V W = -\frac{\langle V, W \rangle}{f} \text{grad } f + \widetilde{\nabla_V^F W}$, where $\widetilde{\nabla_V^F W}$ is the lift of $\nabla_V^F W$.

□

Proposition 3.8 (cf. [3], p. 209–210). *Let $B \times_f F$ be a warped product, and R_B, R_F the lifts of the Riemann curvature tensors on B and F . Let be the vector fields $X, Y, Z \in \mathfrak{L}(B \times F, B)$ and $U, V, W \in \mathfrak{L}(B \times F, F)$, and let H^f be the Hessian of f , $H^f(X, Y) = \langle \nabla_X(\text{grad } f), Y \rangle_B$. Then*

1. $R(X, Y)Z \in \mathfrak{L}(B \times F, B)$ is the lift of $R_B(X, Y)Z$.
2. $R(V, X)Y = -\frac{H^f(X, Y)}{f} V$.
3. $R(X, Y)V = R(V, W)X = 0$.
4. $R(X, V)W = -\frac{\langle V, W \rangle}{f} \nabla_X(\text{grad } f)$.
5. $R(V, W)U = R_F(V, W)U + \frac{\langle \text{grad } f, \text{grad } f \rangle}{f^2} (\langle V, U \rangle W - \langle W, U \rangle V)$.

□

Corollary 3.9 (cf. [3], p. 211). *Let $B \times_f F$ be a warped product, with $\dim F > 1$, and let be the vector fields $X, Y \in \mathfrak{L}(B \times F, B)$ and $V, W \in \mathfrak{L}(B \times F, F)$. Then*

1. $\text{Ric}(X, Y) = \text{Ric}_B(X, Y) + \frac{\dim F}{f} H^f(X, Y).$
2. $\text{Ric}(X, V) = 0.$
3. $\text{Ric}(V, W) = \text{Ric}_F(V, W) + \left(\frac{\Delta f}{f} + (\dim F - 1) \frac{\langle \text{grad } f, \text{grad } f \rangle}{f^2} \right) \langle V, W \rangle.$

□

Corollary 3.10 (cf. [3], p. 211). *Let $B \times_f F$ be a warped product, with $\dim F > 1$. Then, the scalar curvature s of $B \times_f F$ is related to the scalar curvatures s_B and s_F of B and F by*

$$s = s_B + \frac{s_F}{f^2} + 2 \dim F \frac{\Delta f}{f} + \dim F (\dim F - 1) \frac{\langle \text{grad } f, \text{grad } f \rangle_B}{f^2}. \quad (3.5)$$

□

3.5 Warped products of semi-regular semi-Riemannian manifolds

In the following we will provide the condition for a degenerate warped product of semi-regular semi-Riemannian manifolds to be a semi-regular semi-Riemannian manifold.

Theorem 3.11. *Let (B, g_B) and (F, g_F) be two radical-stationary semi-Riemannian manifolds, and $f \in \mathcal{F}(B)$ a smooth function so that $df \in \mathcal{A}^\bullet(B)$. Then, the warped product manifold $B \times_f F$ is a radical-stationary semi-Riemannian manifold.*

Proof. We have to show that $\mathcal{K}(X, Y, W) = 0$ for any $X, Y \in \mathfrak{X}(B \times_f F)$ and $W \in \mathfrak{X}_o(B \times_f F)$. It is enough to check this for vector fields which are lifts of vector fields $X_B, Y_B, W_B \in \mathfrak{L}(B \times F, B)$, $X_F, Y_F, W_F \in \mathfrak{L}(B \times F, F)$, where $W_B, W_F \in \mathfrak{X}_o(B \times_f F)$. Then, from the Proposition 3.6:

1. $\mathcal{K}(X_B, Y_B, W_B) = \mathcal{K}_B(X_B, Y_B, W_B) = 0,$
2. $\mathcal{K}(X_B, Y_B, W_F) = \mathcal{K}(X_B, Y_F, W_B) = \mathcal{K}(X_F, Y_B, W_B) = 0,$
3. $\mathcal{K}(X_B, Y_F, W_F) = \mathcal{K}(Y_F, X_B, W_F) = f \langle Y_F, W_F \rangle_F X_B(f) = 0$, because $\langle Y_F, W_F \rangle_F = 0$, and
 $\mathcal{K}(X_F, Y_F, W_B) = -f \langle X_F, Y_F \rangle_F W_B(f) = 0$, from $W_B(f) = 0,$
4. $\mathcal{K}(X_F, Y_F, W_F) = f^2 \mathcal{K}_F(X_F, Y_F, W_F) = 0.$

□

Theorem 3.12. *Let (B, g_B) and (F, g_F) be two semi-regular semi-Riemannian manifolds, and $f \in \mathcal{F}(B)$ a smooth function so that $df \in \mathcal{A}^{\bullet 1}(B)$. Then, the warped product manifold $B \times_f F$ is a semi-regular semi-Riemannian manifold.*

Proof. All contractions of the form $\mathcal{K}(X, Y, \bullet)\mathcal{K}(Z, T, \bullet)$ are well defined, according to Theorem 3.11. From Proposition 2.57, it is enough to show that they are smooth. It is enough to check this for vector fields which are lifts of vector fields $X_B, Y_B, Z_B, T_B \in \mathfrak{L}(B \times F, B)$, $X_F, Y_F, Z_F, T_F \in \mathfrak{L}(B \times F, F)$. Let's denote by \bullet_B and \bullet_F the symbol for the covariant contraction on B , respectively F . Then, from the Proposition 3.6:

$$\begin{aligned}
 \mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_B, T_B, \bullet) &= \mathcal{K}(X_B, Y_B, \bullet_B)\mathcal{K}(Z_B, T_B, \bullet_B) \\
 &\quad + \mathcal{K}(X_B, Y_B, \bullet_F)\mathcal{K}(Z_B, T_B, \bullet_F) \\
 &= \mathcal{K}_B(X_B, Y_B, \bullet_B)\mathcal{K}_B(Z_B, T_B, \bullet_B) \\
 &\in \mathcal{F}(B \times_f F). \\
 \mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_F, T_F, \bullet) &= \mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_B, T_F, \bullet) \\
 &= \mathcal{K}(X_B, Y_B, \bullet_B)\mathcal{K}(Z_B, T_F, \bullet_B) \\
 &\quad + \mathcal{K}(X_B, Y_B, \bullet_F)\mathcal{K}(Z_B, T_F, \bullet_F) = 0. \\
 \mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_F, T_F, \bullet) &= \mathcal{K}(X_B, Y_B, \bullet_B)\mathcal{K}(Z_F, T_F, \bullet_B) \\
 &\quad + \mathcal{K}(X_B, Y_B, \bullet_F)\mathcal{K}(Z_F, T_F, \bullet_F) \\
 &= -\mathcal{K}_B(X_B, Y_B, \bullet_B)f\langle Z_F, T_F \rangle_F df(\bullet_B) \\
 &= -f\langle Z_F, T_F \rangle_F(\nabla_{X_B}^B Y_B)(df) \\
 &\in \mathcal{F}(B \times_f F). \\
 \mathcal{K}(X_B, Y_F, \bullet)\mathcal{K}(T_F, Z_B, \bullet) &= \mathcal{K}(X_B, Y_F, \bullet)\mathcal{K}(Z_B, T_F, \bullet) \\
 &= \mathcal{K}(X_B, Y_F, \bullet_B)\mathcal{K}(Z_B, T_F, \bullet_B) \\
 &\quad + \mathcal{K}(X_B, Y_F, \bullet_F)\mathcal{K}(Z_B, T_F, \bullet_F) \\
 &= f\langle Y_F, \bullet_F \rangle_F X_B(f)\mathcal{K}(Z_B, T_F, \bullet_F) \\
 &= f^3 X_B(f)\mathcal{K}_F(Z_B, T_F, Y_F) \\
 &\in \mathcal{F}(B \times_f F). \\
 \mathcal{K}(X_B, Y_F, \bullet)\mathcal{K}(Z_F, T_F, \bullet) &= \mathcal{K}(X_B, Y_F, \bullet_B)\mathcal{K}(Z_F, T_F, \bullet_B) \\
 &\quad + \mathcal{K}(X_B, Y_F, \bullet_F)\mathcal{K}(Z_F, T_F, \bullet_F) \\
 &= f^3 X_B(f)\langle Y_F, \bullet_F \rangle_F \mathcal{K}_F(Z_F, T_F, \bullet_F) \\
 &= f^3 X_B(f)\mathcal{K}_F(Z_F, T_F, Y_F) \\
 &\in \mathcal{F}(B \times_f F).
 \end{aligned}$$

□

Remark 3.13. Even though (B, g_B) and (F, g_F) are non-degenerate semi-Riemannian manifolds, if the function f becomes 0, the warped product manifold $B \times_f F$ is a singular semi-Riemannian manifold.

Corollary 3.14. *Let's consider that (B, g_B) is a non-degenerate semi-Riemannian manifold, and let $f \in \mathcal{F}(B)$. If (F, g_F) is radical-stationary, then the warped product $B \times_f F$ also is radical-stationary. If (F, g_F) is semi-regular, then the warped product $B \times_f F$ also is semi-regular. In particular, if both manifolds (B, g_B) and (F, g_F) are non-degenerate, and the warping function $f \in \mathcal{F}(B)$, then $B \times_f F$ is semi-regular.*

Proof. If the manifold (B, g_B) is non-degenerate, then any function $f \in \mathcal{F}(B)$ also satisfies $df \in \mathcal{A}^\bullet(B)$ and $df \in \mathcal{A}^{\bullet 1}(B)$. Then the corollary follows from Theorems 3.11 and 3.12. \square

Proposition 3.15 (The case $f \equiv 0$). *$B \times_0 F$ is a singular semi-Riemannian manifold with degenerate metric of constant rank $g = \dim B$.*

Proof. The proof can be found in [19], p. 287. In fact, Kupeli does even more in [19], by showing that any radical-stationary semi-Riemannian manifold is locally a warped product of the form $B \times_0 F$. \square

Remark 3.16. The warped product of non-degenerate semi-Riemannian manifolds stays non-degenerate for $f > 0$. If $f \rightarrow 0$, we can see for example from [3] that the connection ∇ ([3], p. 206–207), the Riemann curvature R_∇ ([3], p. 209–210), the Ricci tensor Ric and the scalar curvature s ([3], p. 211) diverge in general.

3.6 Riemann curvature of semi-regular warped products

In this section we will assume (B, g_B) and (F, g_F) to be semi-regular semi-Riemannian manifolds, $f \in \mathcal{F}(B)$ a smooth function so that $df \in \mathcal{A}^{\bullet 1}(B)$, and $B \times_f F$ the warped product of B and F . The central point is to find the relation between the Riemann curvature R of $B \times_f F$ and those on (B, g_B) and (F, g_F) . The relations are similar to those for the non-degenerate case (cf. [3], p. 210–211) for the Riemann curvature operator $R(-, -)$, but since this operator is not well defined and is divergent for degenerate metric, we need to use the Riemann curvature tensor $R(-, -, -, -)$. The proofs given here are based only on formulae which work for the degenerate case as well.

Definition 3.17. Let (M, g) be a semi-regular semi-Riemannian manifold. The *Hessian* of a scalar field f satisfying $df \in \mathcal{A}^{\bullet 1}(M)$ is the smooth tensor field $H^f \in \mathcal{T}_2^0 M$ defined by

$$H^f(X, Y) := (\nabla_X df)(Y) \quad (3.6)$$

for any $X, Y \in \mathfrak{X}(M)$.

Theorem 3.18. *Let $B \times_f F$ be a degenerate warped product of semi-regular semi-Riemannian manifolds with $f \in \mathcal{F}(B)$ a smooth function so that $df \in \mathcal{A}^{\bullet 1}(B)$, and R_B, R_F the lifts of the Riemann curvature tensors on B and F . Let $X, Y, Z, T \in \mathfrak{L}(B \times F, B)$, $U, V, W, Q \in \mathfrak{L}(B \times F, F)$, and let H^f be the Hessian of f (which exists because $df \in \mathcal{A}^{\bullet 1}(B)$, see Definition 3.17). Then:*

1. $R(X, Y, Z, T) = R_B(X, Y, Z, T)$
2. $R(X, Y, Z, Q) = 0$
3. $R(X, Y, W, Q) = 0$
4. $R(U, V, Z, Q) = 0$
5. $R(X, V, W, T) = -fH^f(X, T)\langle V, W \rangle_F$
6. $R(U, V, W, Q) = R_F(U, V, W, Q) + f^2 \langle df, df \rangle_{\bullet B} (\langle U, W \rangle_F \langle V, Q \rangle_F - \langle V, W \rangle_F \langle U, Q \rangle_F)$

the other cases being obtained by the symmetries of the Riemann curvature tensor.

Proof. In order to prove these identities, we will use the Koszul formula for the Riemann curvature from equation (2.78). We will denote the covariant contraction with \bullet on $B \times_f F$, and with $\overset{B}{\bullet}$ and $\overset{F}{\bullet}$ on B , respectively F .

$$\begin{aligned}
 (1) \quad R(X, Y, Z, T) &= X\mathcal{K}(Y, Z, T) - Y\mathcal{K}(X, Z, T) - \mathcal{K}([X, Y], Z, T) \\
 &\quad + \mathcal{K}(X, Z, \bullet)\mathcal{K}(Y, T, \bullet) - \mathcal{K}(Y, Z, \bullet)\mathcal{K}(X, T, \bullet) \\
 &= X\mathcal{K}(Y, Z, T) - Y\mathcal{K}(X, Z, T) - \mathcal{K}([X, Y], Z, T) \\
 &\quad + \mathcal{K}(X, Z, \overset{B}{\bullet})\mathcal{K}(Y, T, \overset{B}{\bullet}) - \mathcal{K}(Y, Z, \overset{B}{\bullet})\mathcal{K}(X, T, \overset{B}{\bullet}) \\
 &= R_B(X, Y, Z, T)
 \end{aligned}$$

where we applied (2) from the Proposition 3.6.

$$\begin{aligned}
 (2) \quad R(X, Y, Z, Q) &= X\mathcal{K}(Y, Z, Q) - Y\mathcal{K}(X, Z, Q) - \mathcal{K}([X, Y], Z, Q) \\
 &\quad + \mathcal{K}(X, Z, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, Z, \bullet)\mathcal{K}(X, Q, \bullet) \\
 &= \mathcal{K}(X, Z, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, Z, \bullet)\mathcal{K}(X, Q, \bullet) \\
 &= \mathcal{K}(X, Z, \overset{B}{\bullet})\mathcal{K}(Y, Q, \overset{B}{\bullet}) - \mathcal{K}(Y, Z, \overset{B}{\bullet})\mathcal{K}(X, Q, \overset{B}{\bullet}) \\
 &= 0
 \end{aligned}$$

by the same property, which also leads to

$$\begin{aligned}
 (3) \quad R(X, Y, W, Q) &= X\mathcal{K}(Y, W, Q) - Y\mathcal{K}(X, W, Q) - \mathcal{K}([X, Y], W, Q) \\
 &\quad + \mathcal{K}(X, W, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, W, \bullet)\mathcal{K}(X, Q, \bullet) \\
 &= \mathcal{K}(X, W, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, W, \bullet)\mathcal{K}(X, Q, \bullet) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad R(U, V, Z, Q) &= U\mathcal{K}(V, Z, Q) - V\mathcal{K}(U, Z, Q) - \mathcal{K}([U, V], Z, Q) \\
 &\quad + \mathcal{K}(U, Z, \bullet)\mathcal{K}(V, Q, \bullet) - \mathcal{K}(V, Z, \bullet)\mathcal{K}(U, Q, \bullet) \\
 &= U(f\langle V, Q \rangle_F Z(f)) - V(f\langle U, Q \rangle_F Z(f)) \\
 &\quad - f\langle [U, V], Q \rangle_F Z(f) \\
 &\quad + \mathcal{K}(U, Z, \bullet^B)\mathcal{K}(V, Q, \bullet^B) - \mathcal{K}(V, Z, \bullet^B)\mathcal{K}(U, Q, \bullet^B) \\
 &\quad + \mathcal{K}(U, Z, \bullet^F)\mathcal{K}(V, Q, \bullet^F) - \mathcal{K}(V, Z, \bullet^F)\mathcal{K}(U, Q, \bullet^F) \\
 &= fZ(f)(U\langle V, Q \rangle_F - V\langle U, Q \rangle_F - \langle [U, V], Q \rangle_F) \\
 &\quad + \mathcal{K}(U, Z, \bullet^F)\mathcal{K}(V, Q, \bullet^F)_F - \mathcal{K}(V, Z, \bullet^F)\mathcal{K}(U, Q, \bullet^F)_F \\
 &= fZ(f)(U\langle V, Q \rangle_F - V\langle U, Q \rangle_F - \langle [U, V], Q \rangle_F) \\
 &\quad + f\langle U, \bullet^F \rangle_F Z(f)\mathcal{K}(V, Q, \bullet^F)_F \\
 &\quad - f\langle V, \bullet^F \rangle_F Z(f)\mathcal{K}(U, Q, \bullet^F)_F \\
 &= fZ(f)(U\langle V, Q \rangle_F - V\langle U, Q \rangle_F - \langle [U, V], Q \rangle_F) \\
 &\quad + \mathcal{K}(V, Q, U)_F - \mathcal{K}(U, Q, V)_F \\
 &= 0
 \end{aligned}$$

where we used (3) and (4) from the Proposition 3.6, together with the Definition 2.29. We also used the property that the covariant contraction on F cancels the coefficient f^2 of $\mathcal{K}(U, V, W)_F$.

$$\begin{aligned}
 (5) \quad R(X, V, W, T) &= X\mathcal{K}(V, W, T) - V\mathcal{K}(X, W, T) - \mathcal{K}([X, V], W, T) \\
 &\quad + \mathcal{K}(X, W, \bullet)\mathcal{K}(V, T, \bullet) - \mathcal{K}(V, W, \bullet)\mathcal{K}(X, T, \bullet) \\
 &= -X(fT(f)\langle V, W \rangle_F) \\
 &\quad - \mathcal{K}(V, W, \frac{B}{\bullet})\mathcal{K}(X, T, \frac{B}{\bullet}) \\
 &\quad + \mathcal{K}(X, W, \frac{F}{\bullet})\mathcal{K}(V, T, \frac{F}{\bullet})_F \\
 &= -X(fT(f)\langle V, W \rangle_F) \\
 &\quad + f\langle V, W \rangle_F df(\bullet)\mathcal{K}(X, T, \frac{B}{\bullet})_B \\
 &\quad + X(f)\langle W, \frac{F}{\bullet} \rangle_F T(f)\langle V, \frac{F}{\bullet} \rangle_F \\
 &= -X(f)T(f)\langle V, W \rangle_F - fX(T(f))\langle V, W \rangle_F \\
 &\quad + f\langle V, W \rangle_F \mathcal{K}(X, T, \frac{B}{\bullet})_B df(\frac{B}{\bullet}) \\
 &\quad + X(f)T(f)\langle W, V \rangle_F \\
 &= f\langle V, W \rangle_F \left[\mathcal{K}(X, T, \frac{B}{\bullet})_B df(\frac{B}{\bullet}) - X(T(f)) \right] \\
 &= f\langle V, W \rangle_F \left[\mathcal{K}(X, T, \frac{B}{\bullet})_B df(\frac{B}{\bullet}) - X\langle T, \text{grad } f \rangle_B \right] \\
 &= -fH^f(X, T)\langle V, W \rangle_F
 \end{aligned}$$

where we applied the definition of the Hessian for semi-regular semi-Riemannian manifolds, for f so that $df \in \mathcal{A}^{\bullet 1}(B)$, and the properties of the Koszul derivative of warped products, as in the Proposition 3.6.

$$\begin{aligned}
 (6) \quad R(U, V, W, Q) &= U\mathcal{K}(V, W, Q) - V\mathcal{K}(U, W, Q) - \mathcal{K}([U, V], W, Q) \\
 &\quad + \mathcal{K}(U, W, \bullet)\mathcal{K}(V, Q, \bullet) - \mathcal{K}(V, W, \bullet)\mathcal{K}(U, Q, \bullet) \\
 &= R_F(U, V, W, Q) \\
 &\quad + \mathcal{K}(U, W, \frac{B}{\bullet})\mathcal{K}(V, Q, \frac{B}{\bullet}) - \mathcal{K}(V, W, \frac{B}{\bullet})\mathcal{K}(U, Q, \frac{B}{\bullet}) \\
 &= R_F(U, V, W, Q) \\
 &\quad + f^2\langle U, W \rangle_F df(\frac{B}{\bullet})\langle V, Q \rangle_F df(\frac{B}{\bullet}) \\
 &\quad - f^2\langle V, W \rangle_F df(\frac{B}{\bullet})\langle U, Q \rangle_F df(\frac{B}{\bullet}) \\
 &= R_F(U, V, W, Q) \\
 &\quad + f^2\langle df, df \rangle_{\bullet B} (\langle U, W \rangle_F \langle V, Q \rangle_F \\
 &\quad - \langle V, W \rangle_F \langle U, Q \rangle_F)
 \end{aligned}$$

□

Remark 3.19. Despite the fact that the Riemann tensor $R(-, -)$ is divergent when the warping function converges to 0 even for warped products of non-degenerate metrics ([3], p. 209–210), Theorem 3.18 shows again that the Riemann curvature tensor $R(-, -, -, -)$ is smooth.

3.7 Polar and spherical warped products

In the following, we use the degenerate inner product of semi-regular manifolds to construct other manifolds. We start by providing a recipe to obtain from warped products spherical solutions of various dimension.

3.7.1 Polar warped products

Let $\mu, \rho \in \mathcal{F}(\mathbb{R})$ be smooth real functions so that $\mu^2(-r) = \mu^2(r)$ and $\rho^2(-r) = \rho^2(r)$ for any $r \in \mathbb{R}$, $i \in \{1, 2\}$. We can construct the following warped products between the spaces $(\mathbb{R}, \pm\mu^2 dr \otimes dr)$ and S^1 :

$$(\mathbb{R} \times_r S^1, \pm\mu^2 dr \otimes dr + \rho^2 d\vartheta \otimes d\vartheta). \quad (3.7)$$

We define on $\mathbb{R} \times_r S^1$ the equivalence relation $(r_1, \vartheta_1) \sim (r_2, \vartheta_2)$ if and only if either $r_1 = r_2$ and $\vartheta_1 = \vartheta_2$, or $r_1 = -r_2$ and $\vartheta_1 \equiv (\vartheta_2 + \pi) \pmod{2\pi}$.

Definition 3.20. The manifold $(M, g) := (\mathbb{R}, \pm\mu^2 dr \otimes dr) \times_\rho S^1 / \sim$ is named the *polar warped product* between $(\mathbb{R}, \pm\mu^2 dr \otimes dr)$ and S^1 .

The manifold M is diffeomorphic to \mathbb{R}^2 .

We are looking for conditions which ensure the smoothness of the metric g on M .

Proposition 3.21. *The metric g on M is smooth if and only if the following limit exists and is smooth:*

$$\lim_{r \rightarrow 0} \frac{\pm\mu^2 r^2 - \rho^2}{r^4}. \quad (3.8)$$

Proof. The metric on $\mathbb{R}^2 - \{(0, 0)\}$ is, in Cartesian coordinates:

$$\begin{aligned} g &= \frac{1}{r^2} \begin{pmatrix} r \cos \vartheta & -\sin \vartheta \\ r \sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} \pm\mu^2 & 0 \\ 0 & \rho^2 \end{pmatrix} \begin{pmatrix} r \cos \vartheta & r \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \\ &= \frac{1}{r^2} \begin{pmatrix} \pm\mu^2 r^2 \cos^2 \vartheta + \rho^2 \sin^2 \vartheta & (\pm\mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta \\ (\pm\mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta & \pm\mu^2 r^2 \sin^2 \vartheta + \rho^2 \cos^2 \vartheta \end{pmatrix} \\ &= \frac{1}{r^2} \begin{pmatrix} \pm\mu^2 r^2 - (\pm\mu^2 r^2 - \rho^2) \sin^2 \vartheta & (\pm\mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta \\ (\pm\mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta & \pm\mu^2 r^2 - (\pm\mu^2 r^2 - \rho^2) \cos^2 \vartheta \end{pmatrix} \\ &= \frac{1}{r^2} \begin{pmatrix} \pm\mu^2 r^2 - \frac{\pm\mu^2 r^2 - \rho^2}{r^2} y^2 & \frac{\pm\mu^2 r^2 - \rho^2}{r^2} xy \\ \frac{\pm\mu^2 r^2 - \rho^2}{r^2} xy & \pm\mu^2 r^2 - \frac{\pm\mu^2 r^2 - \rho^2}{r^2} x^2 \end{pmatrix} \end{aligned} \quad (3.9)$$

Hence, g is smooth if and only if the limit (3.8) exists and is smooth. \square

Remark 3.22. The smoothness of g on M is ensured by the condition that $\rho^2(r) = \pm\mu^2r^2 + u(r)r^4$ for some smooth function $u : \mathbb{R} \rightarrow \mathbb{R}$.

The metric becomes, in Cartesian coordinates,

$$g = \begin{pmatrix} \pm\mu^2 + uy^2 & -uxy \\ -uxy & \pm\mu^2 + ux^2 \end{pmatrix} \quad (3.10)$$

The determinant of the metric is

$$\det g = \mu^4 \pm u\mu^2r^2, \quad (3.11)$$

and it follows that the metric becomes degenerate if $\mu = 0$ or $\mu^2 = \pm ur^2$.

Remark 3.23. If we want the metric to be semi-regular, we need to make sure that the equation (2.88) is respected. Since the coefficients μ and ρ depend only on r , it suffices that $\text{supp}(\partial_r\mu) \subseteq \text{supp}(\mu)$ and that there exists a smooth function $f \in \mathcal{F}(\mathbb{R})$ so that $\text{supp}(f) \subseteq \text{supp}(\mu)$ and

$$\frac{\partial\rho^2(r)}{\partial r} = f(r)\mu(r). \quad (3.12)$$

The next example shows how we can obtain the Euclidean plane \mathbb{R}^2 from a degenerate warped product.

Example 3.1. *The flat metric on $\mathbb{R}^2 - \{(0,0)\}$ can be expressed in polar coordinates (r, ϑ) as*

$$g = dr \otimes dr + r^2 d\vartheta \otimes d\vartheta. \quad (3.13)$$

The manifold $\mathbb{R}^2 - \{(0,0)\}$ can be obtained as the non-degenerate warped product $\mathbb{R}^+ \times_r S^1$, where $\mathbb{R}^+ = (0, \infty)$, with the natural metric dr^2 , and S^1 is the unit circle parameterized by ϑ , with the metric $d\vartheta^2$. The metric of $\mathbb{R}^+ \times_r S^1$ becomes degenerate at the point $r = 0$. We can use the degenerate warped product $\mathbb{R} \times_r S^1$, where the metric has the same form as in equation (3.13), and obtain a cylinder whose metric becomes degenerate at the points $r = 0$. The coordinate r is allowed here to become 0 or negative. The polar warped product $M = \mathbb{R} \times_r S^1 / \sim$ is isometric to the Euclidean space \mathbb{R}^2 .

The following example shows how we can obtain the sphere S^2 from a degenerate warped product.

Example 3.2. *Let's rename the coordinate r to φ , let's take instead of $\rho(r)$ the function $\sin \varphi$, and let's make the metric on \mathbb{R} to be $d\varphi \otimes d\varphi$ (hence $\mu^2(\varphi) = 1$). Since $\sin \varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots$, it follows that $\sin \varphi = \varphi - \varphi^3 h(\varphi)$, where h is a smooth function. Hence, $\sin^2 \varphi = \varphi^2 + u(\varphi)\varphi^4$, where $u(\varphi) = -2h(\varphi) + \varphi^2 h^2(\varphi)$ is a smooth function,*

and the smoothness of the metric g at $(0,0)$ is ensured. Let us now use instead the equivalence relation from §3.7.1, the relation defined by $(\varphi_1, \vartheta_1) \sim (\varphi_2, \vartheta_2)$ if and only if either $\varphi_1 \equiv \varphi_2 \pmod{2\pi}$ and $\vartheta_1 = \vartheta_2$, or $\varphi_1 \equiv -\varphi_2 \pmod{2\pi}$ and $\vartheta_1 \equiv (\vartheta_2 + \pi) \pmod{2\pi}$. We obtain the sphere $S^2 \cong \mathbb{R} \times_{\sin \varphi} S^1 / \sim$, having the metric

$$g_{S^2} = d\varphi \otimes d\varphi + \sin^2 \varphi d\vartheta \otimes d\vartheta. \quad (3.14)$$

The usual spherical coordinates can be obtained by restraining the coordinates (ϑ, φ) to the domain $[0, 2\pi) \times [0, \pi]$.

3.7.2 Spherical warped products

In a similar manner as in §3.7.1, we can define *spherical warped products*. We will work on $\mathbb{R} \times_{\rho} S^2$, where the sphere S^2 has the metric and parameterization as in the Example 3.2. The equivalence relation is defined as $(r_1, \vartheta_1, \varphi_1) \sim (r_2, \vartheta_2, \varphi_2)$ if and only if either $r_1 = r_2$ and $\vartheta_1 = \vartheta_2$ and $\varphi_1 = \varphi_2$, or $r_1 = -r_2$ and $\vartheta_1 \equiv (\vartheta_2 + \pi) \pmod{2\pi}$ and $\varphi_1 = \varphi_2$. We start with real smooth functions $\mu, \rho \in \mathcal{F}(\mathbb{R})$ so that $\mu^2(-r) = \mu^2(r)$ and $\rho^2(-r) = \rho^2(r)$ for any $r \in \mathbb{R}$, $i \in \{1, 2\}$, exactly as in the polar case. We can construct the following warped products, between the spaces $(\mathbb{R}, \pm \mu^2 dr \otimes dr)$ and S^2 :

$$(\mathbb{R} \times_{\rho} S^2, \pm \mu^2 dr \otimes dr + \rho^2 (d\varphi \otimes d\varphi + \sin^2 \varphi d\vartheta \otimes d\vartheta)). \quad (3.15)$$

Let $(M, g) = (\mathbb{R}, \pm \mu^2 dr \otimes dr) \times_{\rho} S^2 / \sim$. The manifold is $M = \mathbb{R}^3$. From §3.7.1 it follows that for any plane of M containing the axis $\mathbb{R} \times (0,0)$ the smoothness results from the condition $\rho^2(r) = \pm \mu^2 r^2 + u(r)r^4$ for some function $u : \mathbb{R} \rightarrow \mathbb{R}$. The smoothness of g in these planes ensures its smoothness on the entire M . Moreover, by similar considerations it follows that M is semi-regular from the same condition given by the equation (3.12).

The same method can be used to obtain *n-spherical warped products*, by factoring the warped product $\mathbb{R} \times_{\rho} S^n$.

Example 3.3. As a direct application we can obtain the Euclidean space \mathbb{R}^3 in spherical coordinates from the degenerate warped product $\mathbb{R} \times_r S^2$.

Example 3.4. Similar to the Example 3.2, we can define an equivalence \sim so that the 3-sphere S^3 can be obtained as the spherical warped product $S^3 \cong \mathbb{R} \times_{\sin \gamma} S^2 / \sim$, having the metric

$$g_{S^3} = d\gamma \otimes d\gamma + \sin^2 \gamma (d\varphi \otimes d\varphi + \sin^2 \varphi d\vartheta \otimes d\vartheta). \quad (3.16)$$

Example 3.5. If in equation (3.14) we replace $\sin^2 \gamma$ with $\sinh^2 \gamma$, and \sim with the equivalence relation defined at the beginning of this section, we obtain the hyperbolic 3-space H^3 of constant sectional curvature -1 .

3.8 Applications of semi-regular warped products

From the viewpoint of singular semi-Riemannian geometry, the semi-regular warped products provide a way to construct semi-regular manifolds. This method, together with those presented in section §2.9, allow the construction of a large number of semi-regular singularities.

As applications to cosmology, the Friedmann-Lemaître-Robertson-Walker spacetime, studied in Chapter 6, is a semi-regular warped product. Hence, it is semi-regular (section §6.1). But it is more than this, it is quasi-regular (section §6.2).

The standard stationary black holes have singularities whose metric is singular. In the usual coordinates, there are components of the metric tensor which blow up. But maybe these coordinates are singular, and in other, regular coordinates, the metric appears non-singular. This has been shown to be true for the apparent singularity on the event horizon, in the Eddington-Finkelstein coordinates [65, 66]. But the singularity inside the black hole is genuine, and the metric cannot be made regular by a coordinate transformation. But the metric can be made analytic for all the Schwarzschild, Reissner-Nordström, and Kerr-Newman singularities (see Chapter 7). In the case of the Schwarzschild singularity, it can even be made semi-regular. The metric can be viewed as a degenerate warped product metric.

Chapter 4

Cartan's structural equations for degenerate metric

This chapter contains text from author's paper [76].

Cartan's structural equations show in a compact way the relation between a connection and its curvature, and reveals their geometric interpretation in terms of moving frames. On singular semi-Riemannian manifolds, because the metric is allowed to be degenerate, there are some obstructions in constructing the geometric objects normally associated to the metric. We can no longer construct local orthonormal frames and coframes, or define a metric connection and its curvature operator. But we will see that if the metric is radical stationary, we can construct objects similar to the connection and curvature forms of Cartan, to which they reduce if the metric is non-degenerate. We write analogs of Cartan's first and second structural equations. As a byproduct we will find a compact version of the Koszul formula.

4.1 Introduction

In Riemannian and semi-Riemannian geometry, Cartan's first and second structural equations establish the relation between a local orthonormal frame, the connection, and its curvature. But in singular (semi-)Riemannian geometry, we cannot invert the metric to construct orthonormal coframes, there is no Levi-Civita connection, and no curvature operator. One important operation is the contraction between covariant indices, which requires the inverse of the metric tensor, absent in the degenerate case.

These problems were avoided in [67], where instead of the metric connection was used the Koszul form, and it was defined a Riemann curvature $R(-, -, -, -)$, which coincides to

the usual Riemann curvature tensor if the metric is non-degenerate. As it was shown there, the covariant contraction at a point $p \in M$ can be defined only on the subspace of the cotangent space which consists on covectors $\omega \in T_p^*M$ which are of the form $\omega(V) = \langle U, V \rangle$, $U, V \in T_pM$. The contraction was shown to be well defined and has been extended to tensors of higher order. This contraction was used to define the Riemann curvature tensor $R(-, -, -, -)$.

In this chapter, I will show how to extend Cartan's formalism to singular semi-Riemannian geometry.

4.2 The first structural equation

Cartan's first structural equation shows how a moving coframe rotates when moving in one direction, due to the connection. In the following, we will derive the first structural equation for the case when the metric is allowed to be degenerate. Of course, in this case we will not have a notion of local orthonormal frame, and we will work instead with vectors and annihilator covectors. The following decomposition of the Koszul form will be needed to derive the first structural equation.

4.2.1 The decomposition of the Koszul form

Lemma 4.1.

$$2\mathcal{K}(X, Y, Z) = (dY^\flat)(X, Z) + (\mathcal{L}_Y g)(X, Z). \quad (4.1)$$

Proof. From the formula for the exterior derivative we get:

$$\begin{aligned} (dY^\flat)(X, Z) &= X(Y^\flat(Z)) - Z(Y^\flat(X)) - Y^\flat([X, Z]) \\ &= X\langle Y, Z \rangle - Z\langle X, Y \rangle + \langle Y, [Z, X] \rangle. \end{aligned}$$

The Lie derivative is

$$\begin{aligned} (\mathcal{L}_Y g)(Z, X) &= Yg(Z, X) - g([Y, Z], X) - g(Z, [Y, X]) \\ &= Y\langle Z, X \rangle - \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle. \end{aligned}$$

The equation (4.1) follows then immediately. \square

Corollary 4.2. *To the properties of the Koszul form from Theorem 2.31 we can add the following:*

$$(dY^\flat)(X, Z) = \mathcal{K}(X, Y, Z) - \mathcal{K}(Z, Y, X). \quad (4.2)$$

Proof. This is an immediate consequence of the Lemma 4.1. \square

4.2.2 The connection forms

If $(E_a)_{a=1}^n$ is an orthonormal frame on a non-degenerate semi-Riemannian manifold, then its dual $(\omega^b)_{b=1}^n$ is also orthonormal. The *connection forms* (cf. e.g. [77]) are the 1-forms $\omega_a{}^b$, $1 \leq a, b \leq n$ defined as

$$\omega_a{}^b(X) := \omega^b(\nabla_X E_a). \quad (4.3)$$

It is important to be aware that the indices a, b label the connection one-forms $\omega_a{}^b$, and they don't represent the components of a form.

For general (possibly degenerate) metrics, there is no Levi-Civita connection $\nabla_X E_a$. Also, a frame $(E_a)_{a=1}^n$ cannot be orthonormal, only orthogonal, and its dual $(\omega^b)_{b=1}^n$ cannot be orthogonal, because the metric $\langle\langle \omega, \tau \rangle\rangle_\bullet = g_\bullet(\omega, \tau)$ is not defined for the entire T^*M , but only for $T^\bullet M$. Therefore, we need to find another way to define the connection one-forms.

Definition 4.3. Let $X, Y \in \mathfrak{X}(M)$ be two vector fields. Then, the *connection form* associated to X, Y is the one-form defined as

$$\omega_{XY}(Z) := \mathcal{K}(Z, X, Y). \quad (4.4)$$

In particular, we define ω_{ab} by

$$\omega_{ab}(X) := \omega_{E_a E_b}(X). \quad (4.5)$$

Remark 4.4. The fact that ω_{XY} is a one-form follows from the properties (1) and (2) of the Koszul form given in Theorem 2.31.

4.2.3 The first structural equation

Let (M, g) be a radical-stationary manifold (cf. def. 2.38).

Lemma 4.5 (The first structural equation).

$$dX^\flat = \omega_{X\bullet} \wedge \bullet^\flat \quad (4.6)$$

Proof. From the Lemma 4.1 and from the Definition 2.29 of the Koszul form, we have:

$$(\mathrm{d}X^\flat)(Y, Z) = \mathcal{K}(Y, X, Z) - \mathcal{K}(Z, X, Y). \quad (4.7)$$

By replacing the Koszul form with the connection one-form, we get:

$$(\mathrm{d}X^\flat)(Y, Z) = \omega_{XZ}(Y) - \omega_{XY}(Z). \quad (4.8)$$

By using the properties of the covariant contraction and the property of (M, g) of being radical-stationary, we can expand the Koszul form as

$$\mathcal{K}(X, Y, Z) = \mathcal{K}(X, Y, \bullet) \langle \bullet, Z \rangle = \mathcal{K}(X, Y, \bullet) \left(\bullet^\flat(Z) \right). \quad (4.9)$$

We can do the same for the connection one-form:

$$\begin{aligned} \omega_{YZ}(X) &= \omega_{Y\bullet}(X) \langle \bullet, Z \rangle = \omega_{Y\bullet}(X) \left(\bullet^\flat(Z) \right) \\ &= (\omega_{Y\bullet} \otimes \bullet^\flat)(X, Z). \end{aligned} \quad (4.10)$$

The equation (4.8) becomes

$$\begin{aligned} (\mathrm{d}X^\flat)(Y, Z) &= (\omega_{X\bullet} \otimes \bullet^\flat)(Y, Z) - (\omega_{X\bullet} \otimes \bullet^\flat)(Z, Y) \\ &= (\omega_{X\bullet} \wedge \bullet^\flat)(Y, Z). \end{aligned} \quad (4.11)$$

□

The following corollary shows how we get the first structural equation as we know it.

Corollary 4.6. *If the metric g is non-degenerate, $(E_a)_{a=1}^n$ is an orthonormal frame, and $(\omega^a)_{a=1}^n$ is its dual, then*

$$\mathrm{d}\omega^a = -\omega_s^a \wedge \omega^s. \quad (4.12)$$

Proof. We have from Theorem 2.31 (5) that

$$\omega_{E_a E_b}(X) + \omega_{E_b E_a}(X) = X \langle E_a, E_b \rangle = X(\delta_{ab}) = 0, \quad (4.13)$$

and therefore

$$\omega_{E_a E_b} = -\omega_{E_b E_a}. \quad (4.14)$$

From equation (4.6) we obtain:

$$\mathrm{d}E_a^\flat = \omega_{E_a E_s} \wedge \omega^s. \quad (4.15)$$

Since $\omega_{E_a E_s} = -\omega_{E_s E_a}$ and $\omega^a = E_a^\flat$, the equation (4.12) follows. □

Remark 4.7. The version of the first structural equation obtained here has the advantage that it can be defined for general vector fields, which are not necessarily from an orthonormal local frame, or a local frame in general. It is well defined even if the metric becomes degenerate (but radical-stationary). Of course, at the points where the signature changes we should not expect to have continuity, but on the regions of constant signature the contraction is smooth. If the manifold (M, g) is semi-regular, the smoothness is ensured even at the points where the metric changes its signature.

4.3 The second structural equation

4.3.1 The curvature forms

Definition 4.8. Let (M, g) be a radical-stationary singular semi-Riemannian manifold, and let $X, Y, Z, T \in \mathfrak{X}(M)$ be four vector fields. Then, the *curvature form* associated to X, Y is defined as

$$\Omega_{XY}(Z, T) := R(X, Y, Z, T). \quad (4.16)$$

In particular, if $(E_a)_{a=1}^n$ is a frame field, we define Ω_{ab} by

$$\Omega_{ab}(Z, T) := \Omega_{E_a E_b}(Z, T). \quad (4.17)$$

4.3.2 The second structural equation

Lemma 4.9 (The second structural equation). *Let (M, g) be a radical-stationary singular semi-Riemannian manifold, and let $X, Y \in \mathfrak{X}(M)$ be two vector fields. Then*

$$\Omega_{XY} = d\omega_{XY} + \omega_{X\bullet} \wedge \omega_{Y\bullet}. \quad (4.18)$$

Proof. From the definition of the exterior derivative it follows that

$$\begin{aligned} d\omega_{XY}(Z, T) &= Z(\omega_{XY}(T)) - T(\omega_{XY}(Z)) - \omega_{XY}([T, Z]) \\ &= Z\mathcal{K}(T, X, Y) - T\mathcal{K}(Z, X, Y) - \mathcal{K}([T, Z], X, Y). \end{aligned} \quad (4.19)$$

On the other hand,

$$\begin{aligned} (\omega_{X\bullet} \wedge \omega_{Y\bullet})(Z, T) &= \omega_{X\bullet}(Z)\omega_{Y\bullet}(T) - \omega_{X\bullet}(T)\omega_{Y\bullet}(Z) \\ &= \mathcal{K}(Z, X, \bullet)\mathcal{K}(T, Y, \bullet) - \mathcal{K}(T, X, \bullet)\mathcal{K}(Z, Y, \bullet). \end{aligned} \quad (4.20)$$

From the equation (2.78), it follows that

$$\begin{aligned} R(X, Y, Z, T) = & Z\mathcal{K}(T, X, Y) - T\mathcal{K}(Z, X, Y) - \mathcal{K}([Z, T], X, Y) \\ & + \mathcal{K}(Z, X, \bullet)\mathcal{K}(T, Y, \bullet) - \mathcal{K}(T, X, \bullet)\mathcal{K}(Z, Y, \bullet), \end{aligned} \quad (4.21)$$

and from the identities (4.19), (4.20) and (4.21) the equation (4.18) follows. \square

Part II

Singular general relativity

Chapter 5

Einstein equation at singularities

5.1 Einstein equation at semi-regular singularities

This chapter contains text from author's paper [67]. We apply these results from chapter 2 to construct a version of Einstein's tensor whose density of weight 2 remains smooth even in the presence of semi-regular singularities. We can thus write a densitized version of Einstein's equation, which is smooth, and which is equivalent to the standard Einstein equation if the metric is non-degenerate.

Section §5.1.2 applies the results from Chapter 2 to General Relativity. This section studies the Einstein's equation on semi-regular semi-Riemannian manifolds. It proposes a densitized version of this equation, which remains smooth on semi-regular spacetimes, and reduces to the standard Einstein equation if the metric is non-degenerate.

5.1.1 The problem of singularities

In 1965 Roger Penrose [42], and later he and S. Hawking [5, 43–46], proved a set of *singularity theorems*. These theorems state that under reasonable conditions the space-time turns out to be *geodesic incomplete* – *i.e.* it has *singularities*. Consequently, some researchers proclaimed that General Relativity predicts its own breakdown, by predicting the singularities [46, 48, 78–81]. Hawking's discovery of the black hole evaporation, leading to his *information loss paradox* [78, 82], made the things even worse. The singularities seem to destroy information, in particular violating the unitary evolution of quantum systems. The reason is that the field equations cannot be continued through singularities.

By applying the results presented in this chapter we shall see that, at least for semi-regular semi-Riemannian manifolds, we can extend Einstein's equation through the singularities. Einstein's equation is replaced by a densitized version which is equivalent to the standard version if the metric is non-degenerate. This equation remains smooth at singularities, which now become harmless.

5.1.2 Einstein's equation on semi-regular spacetimes

To define the Einstein tensor on a semi-regular semi-Riemannian manifold, we normally make use of the Ricci tensor and the scalar curvature:

$$G := \text{Ric} - \frac{1}{2}sg \quad (5.1)$$

These two quantities can be defined even for a degenerate metric, so long as the metric doesn't change its signature (see §2.7.3), but at the points where the signature changes, they can become infinite.

Definition 5.1. A *semi-regular spacetime* is a four-dimensional semi-regular semi-Riemannian manifold having the signature $(0, 3, 1)$ at the points where it is non-degenerate.

Theorem 5.2. Let (M, g) be a semi-regular spacetime. Then its Einstein density tensor of weight 2, $G \det g$, is smooth.

Proof. At the points p where the metric is non-degenerate, the Einstein tensor (5.1) can be expressed using the Hodge $*$ operator by:

$$G_{ab} = g^{st}(*R*)_{asbt}, \quad (5.2)$$

where $(*R*)_{abcd}$ is obtained by taking the Hodge dual of R_{abcd} with respect to the first and the second pairs of indices (cf. e.g. [83], p. 234). Explicitly, if we write the components of the volume form associated to the metric as ε_{abcd} , we have

$$(*R*)_{abcd} = \varepsilon_{ab}{}^{st} \varepsilon_{cd}{}^{pq} R_{stpq}. \quad (5.3)$$

If we employ coordinates, the volume form can be expressed in terms of the Levi-Civita symbol by

$$\varepsilon_{abcd} = \epsilon_{abcd} \sqrt{-\det g}. \quad (5.4)$$

We can rewrite the Einstein tensor as

$$G^{ab} = \frac{g_{kl} \epsilon^{akst} \epsilon^{blpq} R_{stpq}}{\det g}, \quad (5.5)$$

If we allow the metric to become degenerate, the Einstein tensor so defined becomes divergent, as it is expected. But the tensor density $G^{ab} \det g$, of weight 2, associated to it remains smooth, and we get

$$G^{ab} \det g = g_{kl} \epsilon^{akst} \epsilon^{blpq} R_{stpq}. \quad (5.6)$$

Since the spacetime is semi-regular, this quantity is indeed smooth, because it is constructed only from the Riemann curvature tensor, which is smooth (see Theorem 2.62), and from the Levi-Civita symbol, which is constant in the particular coordinate system. The determinant of the metric converges to 0 so that it cancels the divergence which normally would appear in G^{ab} . The tensor density $G_{ab} \det g$, being obtained by lowering its indices, is also smooth. \square

Remark 5.3. Because the densitized Einstein tensor $G_{ab} \det g$ is smooth, it follows that the densitized curvature scalar is smooth

$$s \det g = -g_{ab} G^{ab} \det g, \quad (5.7)$$

and so is the densitized Ricci tensor

$$R_{ab} \det g = g_{as} g_{bt} G^{st} \det g + \frac{1}{2} s g_{ab} \det g. \quad (5.8)$$

Remark 5.4. In the context of General Relativity, on a semi-regular spacetime, if T is the stress-energy tensor, we can write the *densitized Einstein equation*:

$$G \det g + \Lambda g \det g = \kappa T \det g, \quad (5.9)$$

or, in coordinates or local frames,

$$G_{ab} \det g + \Lambda g_{ab} \det g = \kappa T_{ab} \det g, \quad (5.10)$$

where $\kappa := \frac{8\pi\mathcal{G}}{c^4}$, with \mathcal{G} and c being Newton's constant and the speed of light.

5.2 Einstein equation at quasi-regular singularities

This chapter contains text from author's paper [84]. Einstein's equation is rewritten in an equivalent form, which remains valid at the singularities in some major cases. These cases include the Schwarzschild singularity (see section §7.1), the Friedmann-Lemaître-Robertson-Walker Big Bang singularity (see section §6.2), isotropic singularities, and a class of warped product singularities. This equation is constructed in terms of the Ricci

part of the Riemann curvature (as the Kulkarni-Nomizu product between Einstein's equation and the metric tensor).

The expanded Einstein equation applies to a subset of semi-regular singularities, named quasi-regular. Quasi-regular singularities have in addition some interesting properties which will be exploited in the following chapters, especially section §6.3 and 8.

5.2.1 Introduction

The singularities in General Relativity cannot be avoided. Einstein's equation leads to them in very general conditions [5, 42–46], and there the time evolution breaks down. Is this a problem of the theory itself, or of the way we formulate it? This chapter proposes a version of Einstein's equation which is equivalent to the standard version at the points of the spacetime where the metric is not singular. But unlike Einstein's equation, in many cases it can be extended at and beyond the singular points.

The *expanded Einstein equation*, and the quasi-regular spacetimes on which it holds, is introduced in section §5.2.2. It is obtained simply by taking the Kulkarni-Nomizu product between Einstein's equation and the metric tensor. In a quasi-regular spacetime, the metric tensor becomes degenerate at singularities, in a way which cancels them and makes the equations smooth.

The situations when the new version of Einstein's equation extends at singularities include isotropic singularities (section §5.2.3.1), and a class of warped product singularities (section §5.2.3.2). It also contains the Schwarzschild singularity (section §5.2.3.4) and the FLRW Big Bang singularity (section §5.2.3.3).

5.2.2 Expanded Einstein equation and quasi-regular spacetimes

5.2.2.1 The expanded Einstein equation

We will write another equation which is equivalent to Einstein's equation whenever the metric tensor g_{ab} is non-degenerate, but is valid also in a class of situations when g_{ab} becomes degenerate and Einstein's tensor is not defined. Later we will see that our version of Einstein's equation remains smooth in various important situations, such as the FLRW Big-Bang singularity, isotropic singularities, and at the singularity of the Schwarzschild black hole.

We introduce the *expanded Einstein equation*

$$(G \circ g)_{abcd} + \Lambda(g \circ g)_{abcd} = \kappa(T \circ g)_{abcd} \quad (5.11)$$

where the operation

$$(h \circ k)_{abcd} := h_{ac}k_{bd} - h_{ad}k_{bc} + h_{bd}k_{ac} - h_{bc}k_{ad} \quad (5.12)$$

is the *Kulkarni-Nomizu product* of two symmetric bilinear forms h and k .

If the metric is non-degenerate, the Einstein equation and its expanded version are equivalent. If the metric becomes degenerate, its inverse becomes singular, and in general the Riemann, Ricci, and scalar curvatures, and consequently the Einstein tensor G_{ab} , blow up. But for some cases, the metric term from the Kulkarni-Nomizu product $G \circ g$ tends to 0 enough to cancel the blow up of the Einstein tensor.

This cancellation allows us to weaken the condition that the metric tensor is non-degenerate, to some cases when it can be degenerate. We will see that these cases include some important singularities.

5.2.2.2 A more explicit form of the expanded Einstein equation

To give a more explicit form of the expanded Einstein equation, we use the *Ricci decomposition* of the Riemann curvature tensor (see *e.g.* [70, 85, 86]).

Let (M, g) be a Riemannian or a semi-Riemannian manifold of dimension n . The Riemann curvature tensor can be decomposed algebraically as

$$R_{abcd} = S_{abcd} + E_{abcd} + C_{abcd}. \quad (5.13)$$

where

$$S_{abcd} = \frac{1}{n(n-1)} R(g \circ g)_{abcd} \quad (5.14)$$

is the scalar part of the Riemann curvature, and

$$E_{abcd} = \frac{1}{n-2} (S \circ g)_{abcd} \quad (5.15)$$

is the *semi-traceless part* of the Riemann curvature. Here

$$S_{ab} := R_{ab} - \frac{1}{n} R g_{ab} \quad (5.16)$$

is the traceless part of the Ricci curvature.

The *Weyl curvature tensor* is defined as the *traceless part* of the Riemann curvature

$$C_{abcd} = R_{abcd} - S_{abcd} - E_{abcd}. \quad (5.17)$$

Let's return to a spacetime of dimension $n = 4$. By using the equations (5.1) and (5.16) we can write the Einstein tensor in terms of the traceless part of the Ricci tensor and the scalar curvature:

$$G_{ab} = S_{ab} - \frac{1}{4}Rg_{ab}. \quad (5.18)$$

We can use this equation to calculate the *expanded Einstein tensor*:

$$\begin{aligned} G_{abcd} &:= (G \circ g)_{abcd} \\ &= (S \circ g)_{abcd} - \frac{1}{4}R(g \circ g)_{abcd} \\ &= 2E_{abcd} - 3S_{abcd}. \end{aligned} \quad (5.19)$$

The expanded Einstein equation takes now the form

$$2E_{abcd} - 3S_{abcd} + \Lambda(g \circ g)_{abcd} = \kappa(T \circ g)_{abcd}. \quad (5.20)$$

5.2.2.3 Quasi-regular spacetimes

We are interested in singular spacetimes on which the expanded Einstein equation (5.11) can be written and is smooth. From (5.20) we see that this requires the smoothness of the tensors E_{abcd} and S_{abcd} .

In addition, we are interested to have the nice properties of the semi-regular spacetimes. As showed in [67], the semi-regular manifolds are a class of singular semi-Riemannian manifolds which are nice for several reasons. First, they allow contraction between covariant indices for an important class of tensors and differential forms. This is in general prohibited by the fact that when the metric tensor g_{ab} becomes degenerate, it doesn't admit a reciprocal g^{ab} . This also prohibits in general the construction of a Levi-Civita connection. On semi-regular manifolds, this can be done for differential forms, and covariant tensors in general. For vector fields we use instead of $\nabla_X Y$, the *Koszul form*, defined as in (2.20). This defines the Levi-Civita connection implicitly by $\langle \nabla_X Y, Z \rangle = \mathcal{K}(X, Y, Z)$ for a non-degenerate metric, but not when the metric becomes degenerate.

In [67] (and here in chapter 2) we define the Riemann curvature R_{abcd} even for non-degenerate metrics, in a way which avoids the undefined $\nabla_X Y$, but relies on the defined and smooth $\mathcal{K}(X, Y, Z)$. To do this, we require that we can define the covariant derivative of the differential 1-form $\mathcal{K}(X, Y, \cdot)$, and that this is smooth. This requirement is equivalent to the requirements that $\mathcal{K}(X, Y, W) = 0$ whenever W becomes degenerate (*i.e.* $\langle W, X \rangle = 0$ for any X), and that the contraction $\mathcal{K}(X, Y, \bullet)\mathcal{K}(Z, T, \bullet)$ is smooth for any local vector fields X, Y, Z, T . A singular semi-Riemannian manifold satisfying this

condition is named *semi-regular manifold*, and its metric is called *semi-regular metric*. A 4-dimensional semi-regular manifold with metric having the signature at each point (r, s, t) , $s \leq 3$, $t \leq 1$, but which is non-degenerate on a dense subset, is called *semi-regular spacetime* [67]. The Riemann curvature R_{abcd} is smooth, since it can then be defined as in (2.81).

In a semi-regular spacetime, since R_{abcd} is smooth, the densitized Einstein tensor $G_{ab} \det g$ is smooth [67], and we can write a densitized version of the Einstein equation (5.10), which is equivalent to the usual version when the metric is non-degenerate.

Although the semi-regular approach is more general, we explored here the quasi-regular one, which is more strict. Consequently, these results are stronger.

Definition 5.5. We say that a semi-regular manifold (M, g_{ab}) is *quasi-regular*, and that g_{ab} is a *quasi-regular metric*, if:

1. g_{ab} is non-degenerate on a subset dense in M
2. the tensors S_{abcd} and E_{abcd} defined at the points where the metric is non-degenerate extend smoothly to the entire manifold M .

If the quasi-regular manifold M is a semi-regular spacetime, we call it *quasi-regular spacetime*.

Remark 5.6. We can see immediately that on an quasi-regular spacetime the expanded Einstein tensor can be extended also at the points where the metric is degenerate, and the extension is smooth. This is in fact the motivation of Definition 5.5.

Remark 5.7. The expanded Einstein equation (5.11) does not necessarily rely on the semi-regularity of the metric. But in the definition of quasi-regular manifolds we preferred to assume the semi-regularity, because it comes with other good properties, such as the smoothness of R_{abcd} .

5.2.3 Examples of quasi-regular spacetimes

Remark 5.8. The quasi-regular spacetimes are more general than the regular ones (with non-degenerate metric). The question is, are they general enough to cover the singularities which plagued General Relativity? In the following we will see that, at least for some relevant cases, the answer is positive.

5.2.3.1 Isotropic singularities

Isotropic singularities occur in conformal rescalings of non-degenerate metrics, when the scaling function cancels. They were extensively studied by Tod [87–92], Claudel & Newman [93], Anzuino & Tod [94, 95]. The following theorem shows that the isotropic singularities are quasi-regular.

Theorem 5.9 (Isotropic singularities). *Let (M, g_{ab}) be a regular spacetime (we assume therefore that the metric g_{ab} is non-degenerate). Then, if $\Omega : M \rightarrow \mathbb{R}$ is a smooth function, which is non-zero on a dense subset of M , the spacetime $(M, \tilde{g}_{ab} := \Omega^2 g_{ab})$ is quasi-regular.*

Proof. We know from [67] that (M, \tilde{g}_{ab}) is semi-regular.

The Ricci and the scalar curvatures take the following forms ([5], p. 42.):

$$\tilde{R}^a_b = \Omega^{-2} R^a_b + 2\Omega^{-1}(\Omega^{-1})_{;bs} g^{as} - \frac{1}{2}\Omega^{-4}(\Omega^2)_{;st} g^{st} \delta^a_b \quad (5.21)$$

$$\tilde{R} = \Omega^{-2} R - 6\Omega^{-3} \Omega_{;st} g^{st} \quad (5.22)$$

where the covariant derivatives correspond to the metric g . From equation (5.21) we have

$$\tilde{R}_{ab} = \Omega^2 g_{as} \tilde{R}^s_b = R_{ab} + 2\Omega(\Omega^{-1})_{;ab} - \frac{1}{2}\Omega^{-2}(\Omega^2)_{;st} g^{st} g_{ab}, \quad (5.23)$$

which tends to infinity when $\Omega \rightarrow 0$. But we are interested to prove the smoothness of the Kulkarni-Nomizu product $\widetilde{\text{Ric}} \circ \tilde{g}$. We notice that the term \tilde{g} contributes with a factor Ω^2 , and it is enough to prove the smoothness of:

$$\Omega^2 \tilde{R}_{ab} = \Omega^2 R_{ab} + 2\Omega^3(\Omega^{-1})_{;ab} - \frac{1}{2}(\Omega^2)_{;st} g^{st} g_{ab}, \quad (5.24)$$

which follows easily from

$$\begin{aligned} \Omega^3(\Omega^{-1})_{;ab} &= \Omega^3((\Omega^{-1})_{;a})_{;b} = \Omega^3(-\Omega^{-2}\Omega_{;a})_{;b} \\ &= \Omega^3(2\Omega^{-3}\Omega_{;b}\Omega_{;a} - \Omega^{-2}\Omega_{;ab}) \\ &= 2\Omega_{;a}\Omega_{;b} - \Omega\Omega_{;ab} \end{aligned} \quad (5.25)$$

Hence, the tensor $\widetilde{\text{Ric}} \circ \tilde{g}$ is smooth. The fact that $\tilde{R}\tilde{g} \circ \tilde{g}$ is smooth follows from the observation that $\tilde{g} \circ \tilde{g}$ contributes with Ω^4 , and the least power in which Ω appears in the expression (5.22) of \tilde{R} is -3 .

From the above follows that \tilde{E}_{abcd} and \tilde{S}_{abcd} are smooth. Hence the spacetime (M, \tilde{g}_{ab}) is quasi-regular. \square

5.2.3.2 Quasi-regular warped products

Another example useful in cosmology is the following, which is a generalization of the warped products (see *e.g.* [3], p. 204), which we gave in Definition 3.1 and Remark 3.2. We will allow the warped function f to become 0 (generalizing the standard definition, where it is not allowed, because leads to degenerate metrics).

Theorem 5.10 (Quasi-regular warped product). *A degenerate warped product $B \times_f F$ with $\dim B = 1$ and $\dim F = 3$ is quasi-regular.*

Proof. From [73] (and chapter 3) we know that $B \times_f F$ is semi-regular.

Let's denote by g_B , g_F and g the metrics on B , F and $B \times_f F$. We know ([3], p. 211) that, for horizontal vector fields $X, Y \in \mathfrak{L}(B \times F, B)$ and vertical vector fields $V, W \in \mathfrak{L}(B \times F, F)$,

1. $\text{Ric}(X, Y) = \text{Ric}_B(X, Y) + \frac{\dim F}{f} H^f(X, Y)$
2. $\text{Ric}(X, V) = 0$
3. $\text{Ric}(V, W) = \text{Ric}_F(V, W) + (f\Delta f + (\dim F - 1)g_B(\text{grad } f, \text{grad } f))g_F(V, W)$

where Δf is the Laplacian, H^f the Hessian, and $\text{grad } f$ the gradient. It follows that $\text{Ric}(X, V)$ and $\text{Ric}(V, W)$ are smooth, but $\text{Ric}(X, Y)$ in general is not, because of the term containing f^{-1} . But since we take $\dim B = 1$, the only terms in the Kulkarni-Nomizu product $\text{Ric} \circ g$ containing $\text{Ric}(X, Y)$ are of the form

$$\text{Ric}(X, Y)g(V, W) = f^2 \text{Ric}(X, Y)g_F(V, W).$$

Hence, $\text{Ric} \circ g$ is smooth.

From the expression of the scalar curvature

$$R = R_B + \frac{R_F}{f^2} + 2 \dim F \frac{\Delta f}{f} + \dim F (\dim F - 1) \frac{g_B(\text{grad } f, \text{grad } f)}{f^2} \quad (5.26)$$

we conclude that S_{abcd} is smooth too. Hence, $B \times_f F$ is quasi-regular. \square

The following example important in cosmology is a direct application of this result.

5.2.3.3 The Friedmann-Lemaître-Robertson-Walker spacetime

In section §6.2, Corollary 6.7, it is shown that the FLRW spacetime, with smooth $a : I \rightarrow \mathbb{R}$, is quasi-regular.

5.2.3.4 Schwarzschild black hole

The Schwarzschild solution describing a black hole of mass m is discussed in section §7.1, where it is shown that the singularity can be made semi-regular. It is also shown in Corollary 7.10 that it is quasi-regular.

Open Problem 5.11. What can we say about the other stationary black hole solutions? In [96] and [97] we showed that there are coordinate transformations which make the Reissner-Nordström metric and the Kerr-Newman metric analytic at the singularity. This is already a big step, because it allows us to foliate with Cauchy hypersurfaces these spacetimes. Is it possible to find coordinate transformations which make them quasi-regular too?

5.2.4 Open Question

We conclude with the following open question:

Open Problem 5.12. Are quasi-regular spacetimes general enough to cover all possible singularities of General Relativity?

Chapter 6

The Big-Bang singularity

This chapter is based on author's papers [98–100].

6.1 Friedmann-Lemaître-Robertson-Walker spacetime is semi-regular

We show that the Big Bang singularity of the Friedmann-Lemaître-Robertson-Walker model does not raise major problems to General Relativity. We prove a theorem showing that the Einstein equation can be written in a non-singular form, which allows the extension of the spacetime before the Big Bang.

These results follow from our research on singular semi-Riemannian geometry and singular General Relativity [67, 73, 76] (which we applied in previous chapters to the black hole singularities [96, 97, 101, 102]).

6.1.1 Introduction

6.1.1.1 The universe

According to the *cosmological principle*, our expanding universe, although it is so complex, can be considered at very large scale homogeneous and isotropic. This is why we can model the universe, at very large scale, by the solution proposed by A. Friedmann [103–105]. This exact solution to Einstein's equation, describing a homogeneous, isotropic universe, is in general called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, due to the rediscovery and contributions made by Georges Lemaître [106], H. P. Robertson [107–109] and A. G. Walker [110].

The FLRW model shows that the universe should be, at a given moment of time, either in expansion, or in contraction. From Hubble's observations, we know that the universe is currently expanding. The FLRW model shows that, long time ago, there was a very high concentration of matter, which exploded in what we call the *Big Bang*. Was the density of matter at the beginning of the universe so high that the Einstein's equation was singular at that moment? This question received an affirmative answer, under general hypotheses and considering General Relativity to be true, in Hawking's singularity theorem [5, 43–46] (which is an application of the reasoning of Penrose for the black hole singularities [42], backwards in time to the past singularity of the Big Bang).

Of course, given that the extreme conditions which were present at the Big Bang are very far from what our observations encountered so far, and our theories managed to extrapolate, we cannot know precisely what happened then. If because some known or unknown quantum effect the energy condition from the hypothesis of the singularity theorem was not obeyed, the singularity might have been avoided, although it was a very high density. One such possibility is explored in the *loop quantum cosmology* [111–113], which leads to a Big Bounce discrete model of the universe.

We will not explore here the possibility that the Big Bang singularity is prevented to exist by quantum or other kind of effects, because we don't have the complete theory which is supposed to unify General Relativity and Quantum Theory. What we will do in the following is to push the limits of General Relativity to see what happens at the Big Bang singularity, in the context of the FLRW model. We will see that the singularities are not a problem, even if we don't modify General Relativity and we don't assume very repulsive forces which prevented the singularity.

One tends in general to regard the singularities arising in General Relativity as an irremediable problem which forces us to abandon this successful theory [46, 48, 78–81]. In fact, we will see that the singularities of the FLRW model are easy to understand and are not fatal to General Relativity. In [67] we presented an approach to extend the semi-Riemannian geometry to the case when the metric can become degenerate. In [73] we applied this theory to the warped products, and provided by this ways to construct examples of singular semi-Riemannian manifolds of this type. We will develop here some ideas suggested in some of the examples presented there, and apply them to the singularities in the FLRW spacetime. We will see that the singularities of the FLRW metric are even simpler than the black hole singularities, which we discussed in [76, 96, 97, 101].

6.1.1.2 The Friedmann-Lemaître-Robertson-Walker model of the universe

Let's consider the 3-space at any moment of time as being modeled, up to a scaling factor, by a three-dimensional Riemannian space (Σ, g_Σ) . The time is represented as an interval $I \subseteq \mathbb{R}$, with the natural metric $-dt^2$. At each moment of time $t \in I$, the space Σ_t is obtained by scaling (Σ, g_Σ) with a scaling factor $a^2(t)$. The scaling factor is therefore given by a function $a : I \rightarrow \mathbb{R}$, named the *warping function*. The FLRW spacetime is the spacetime $I \times \Sigma$ endowed with the metric

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2. \quad (6.1)$$

It is the *warped product* between (Σ, g_Σ) and $(I, -dt^2)$, with the warping function $a : I \rightarrow \mathbb{R}$.

The typical space Σ can be any Riemannian manifold we may need for our cosmological model, but because of the homogeneity and isotropy conditions, it is in general taken to be, at least at large scale, one of the homogeneous spaces S^3 , \mathbb{R}^3 , and H^3 . In this case, the metric on Σ is, in spherical coordinates (r, θ, ϕ) ,

$$d\Sigma^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (6.2)$$

where $k = 1$ for the 3-sphere S^3 , $k = 0$ for the Euclidean space \mathbb{R}^3 , and $k = -1$ for the hyperbolic space H^3 .

6.1.1.3 The Friedman equations

Once we choose the 3-space Σ , the only unknown part of the FLRW metric is the function $a(t)$. To determine it, we have to make some assumptions about the matter in the universe. In general it is assumed, for simplicity, that the universe is filled with a fluid with mass density $\rho(t)$ and pressure density $p(t)$. The density and the pressure are taken to depend on t only, because we assume the universe to be homogeneous and isotropic. The stress-energy tensor is

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}, \quad (6.3)$$

where u^a is the timelike vector field ∂_t , normalized.

From the energy density component of the Einstein equation, one can derive the *Friedmann equation*

$$\rho = \frac{3}{\kappa} \frac{\dot{a}^2 + k}{a^2}, \quad (6.4)$$

where $\kappa := \frac{8\pi\mathcal{G}}{c^4}$ (\mathcal{G} and c being the gravitational constant and the speed of light, which we will consider equal to 1 for now on, by an appropriate choice of measurement units). From the trace of the Einstein equation, we obtain the *acceleration equation*

$$\rho + 3p = -\frac{6}{\kappa} \frac{\ddot{a}}{a}. \quad (6.5)$$

From these two equations we obtain the *fluid equation*, expressing the conservation of mass-energy:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p). \quad (6.6)$$

Let's assume we know the function a . The Friedman equation (6.4) shows that we can uniquely determine ρ from a . The acceleration equation determines p from both a and ρ . Hence, the function a determines uniquely both ρ and p .

From the recent observations on supernovae, we know that the expansion is accelerated, corresponding to the existence of a positive cosmological constant Λ [114, 115]. The Friedmann's equations were expressed here without Λ , but this doesn't reduce the generality, because the equations containing the cosmological constant are equivalent to those without it, by the substitution

$$\begin{cases} \rho & \rightarrow \rho + \kappa^{-1}\Lambda \\ p & \rightarrow p - \kappa^{-1}\Lambda \end{cases} \quad (6.7)$$

Therefore, for simplicity we will continue to ignore Λ in the following, without any loss of generality.

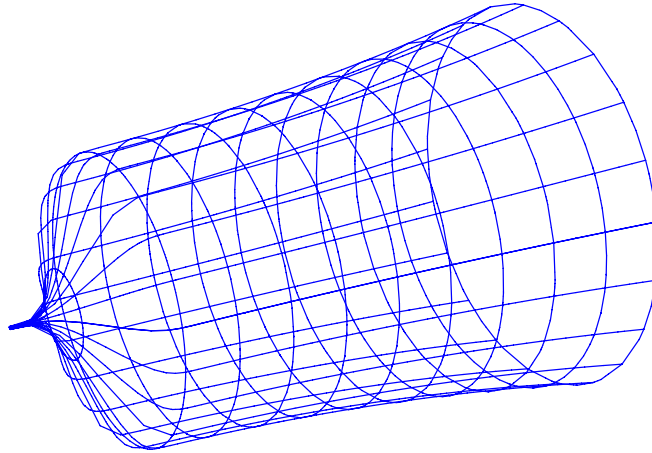


FIGURE 6.1: The standard view is that the universe originated from a very dense state, probably a singularity, and expanded, with a short period of very high acceleration (the inflation).

The standard view on cosmology today is that the universe started with the Big Bang, which is in general assumed to be singular, and then expanded, with a very short period of exponentially accelerated expansion, called *inflation* (Fig. 6.1).

6.1.2 The main ideas

The solution proposed here is simple: to show that the singularities of the FLRW model don't break the evolution equation, we show that the equations can be written in an equivalent form which avoids the infinities in a natural way. We consider useful to prepare the reader with some simple mathematical observations, which will clarify our proof. These observations can be easily understood, and combined they help us understanding the Big Bang singularity in the FLRW spacetime.

6.1.2.1 Distance separation *vs.* topological separation

To understand the singularities in the FLRW model, it is useful to make a parallel with another type of singularities, and their standard resolution in mathematics. Let's consider a surface in \mathbb{R}^3 . In general it can be defined locally as the image of a map $f : U \rightarrow \mathbb{R}^3$, where $U \subset \mathbb{R}^2$ is an open subset of the plane. If the function f is not injective, the surface will have self-intersections. Another way to define the surface is implicitly, as the solution of an equation. In this case it may happen again to have self-intersections. The typical example is the cone, defined as

$$x^2 - y^2 - z^2 = 0. \quad (6.8)$$

We can desingularize it by making the transformation

$$\begin{cases} x &= u \\ y &= uv \\ z &= uw \end{cases} \quad (6.9)$$

which maps the cylinder $v^2 + w^2 = 1$ from the space parametrized by (u, v, w) , to the cone from equation (6.8), in the space parametrized by (x, y, z) . This procedure is very used in mathematics, especially in *algebraic geometry*, and it was studied starting with Isaac Newton [116].

The natural metric on the space (x, y, z) induces, by pull-back, a metric on the cylinder $v^2 + w^2 = 1$ from the space (u, v, w) . The induced metric on the cylinder is singular: the distance between any pair of points of the circle determined by the equations $u = 0$ and $v^2 + w^2 = 1$ is zero. But the points of that circle are distinct.

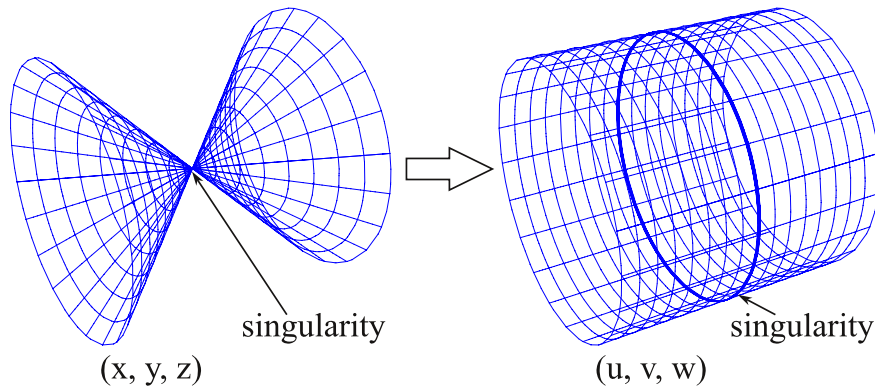


FIGURE 6.2: The old method of resolution of singularities shows how we can “untie” the singularity of a cone and obtain a cylinder. This illustrates the idea that it is not necessary to assume that, at the Big Bang singularity, the entire space was a point, but only that the space metric was 0.

From the viewpoint of the singularities in General Relativity, the main implication is that just because the distance between two points is 0, it doesn’t mean that the two points coincide. We can see something similar even in Special Relativity: the 4-distance between two events separated by a lightlike interval is equal to 0, but those events may be distinct.

6.1.2.2 Degenerate warped product and singularities

The mathematics of General Relativity is a branch of differential geometry, called *semi-Riemannian (or pseudo-Riemannian) geometry* (see e.g. [3]). It is a generalization of the Riemannian geometry, to the case when the metric tensor is still non-degenerate, but its signature is not positive. In this geometric framework are defined notions like contraction, Levi-Civita connection, covariant derivative, Riemann curvature, Ricci tensor, scalar curvature, Einstein tensor. These are the main ingredients of the theory of General Relativity [3, 5, 117].

The problem with the singularities is that there, these main ingredients can’t be defined, or become infinite. The perfection of semi-Riemannian geometry is broken there, and by this, it is usually concluded that the same happens with General Relativity.

In [67] we introduced a way to extend semi-Riemannian geometry to the degenerate case. There is a previous approach [19, 20], which works for metric of constant signature. Our need was to have a theory valid for variable signature (because the metric changes from being non-degenerate to being degenerate), and which in addition allows us to define the Riemann, Ricci and scalar curvatures in an invariant way, and something like the covariant derivative for the differential forms and tensor fields which are of use in General

Relativity. After developing this theory, introduced in [67, 76], we generalized the notion of warped product to the degenerate case, providing by this a way to construct useful examples of singularities of this nice behaved kind [73].

From the mathematics of degenerate warped products it followed that a warped product like that involved in a FLRW metric (equation 6.1) has only singularities which are well behaved, and which allow the extension of General Relativity to those points. At these singularities, the Riemann curvature tensor R_{abcd} is not degenerate, and it is smooth if a is smooth. The Einstein equation can be replaced by a densitized version, which allows the continuation to the singular points and avoids the infinities.

6.1.2.3 What happens if the density becomes infinite?

In the Friedmann equations (6.4), (6.5), and (6.6), the variables are a , the mass/energy density ρ and the pressure density p . When $a \rightarrow 0$, ρ appears to tend to infinity, because a finite amount of matter occupies a volume equal to 0. Similarly, the pressure density p may become infinite. How can we rewrite the equations to avoid the infinities? As it will turn out, not only there is a solution to do this, but the quantities involved are actually the natural ones. As present in the equations, both ρ and p are scalar fields. But the adequate, invariant quantities actually involve the *volume element*, or the *volume form*

$$d_{vol} := \sqrt{-g} dt \wedge dx \wedge dy \wedge dz, \quad (6.10)$$

where by the factor $\sqrt{-g}$ we mean $\sqrt{-\det g_{ab}}$. The densities are in fact not the scalars ρ and p , but the quantities ρd_{vol} and $p d_{vol}$. They are differential 4-forms on the spacetime, and the components of these forms in a coordinate system are $\rho\sqrt{-g}$ and respectively $p\sqrt{-g}$.

Another hint that the natural quantities are the densitized ones is given by the stress-energy tensor. When it is obtained from the Lagrangian, what we actually get is the tensor density

$$T^{ab}\sqrt{-g} = -2 \frac{\delta(\mathcal{L}_M \sqrt{-g})}{\delta g_{ab}} \quad (6.11)$$

The values ρ and p which appear in the Friedmann equations coincide with the components of the corresponding densities only in an orthonormal frame, where the determinant of the metric equals -1 , and we can omit $\sqrt{-g}$. But when $a \rightarrow 0$, an orthonormal frame would become singular, because $\det g \rightarrow 0$. A coordinate system in which the metric has the determinant -1 will necessarily be singular when $a(t) = 0$. In a non-singular coordinate system, $\det g$ has to be variable, as it is in the comoving coordinate system of the FLRW model. From (6.1), the determinant of the metric in the FLRW

coordinates is

$$\det g = -a^6 \det {}_3g_\Sigma, \quad (6.12)$$

where by $\det {}_3g_\Sigma$ we denoted the determinant of the metric tensor of the 3-dimensional typical space Σ . Since the typical space is the same for all moments of time t , $\det {}_3g_\Sigma$ is constant.

Given that the metric's determinant in the comoving coordinates is

$$\sqrt{-g} = a^3 \sqrt{g_\Sigma}, \quad (6.13)$$

which tends to 0 when $a \rightarrow 0$, we see that it might be possible for $\sqrt{-g}$ to cancel the singularity of ρ and p in ρd_{vol} , respectively $p d_{vol}$.

6.1.3 The Big Bang singularity resolution

As explained in section §6.1.2.3, we should account in the mass/energy density and the pressure density for the term $\sqrt{-g}$.

Consequently, we make the following substitution:

$$\begin{cases} \tilde{\rho} = \rho \sqrt{-g} = \rho a^3 \sqrt{g_\Sigma} \\ \tilde{p} = p \sqrt{-g} = p a^3 \sqrt{g_\Sigma} \end{cases} \quad (6.14)$$

We have the following result:

Theorem 6.1. *If a is a smooth function, then the densities $\tilde{\rho}$, \tilde{p} , and the densitized stress-energy tensor $T_{ab}\sqrt{-g}$ are smooth (and therefore nonsingular), even at moments t_0 when $a(t_0) = 0$.*

Proof. The Friedmann equation (6.4) becomes

$$\tilde{\rho} = \frac{3}{\kappa} a (\dot{a}^2 + k) \sqrt{g_\Sigma}, \quad (6.15)$$

from which it follows that if a is a smooth function, $\tilde{\rho}$ is smooth as well.

The acceleration equation (6.5) becomes

$$\tilde{\rho} + 3\tilde{p} = -\frac{6}{\kappa} a^2 \ddot{a} \sqrt{g_\Sigma}, \quad (6.16)$$

which shows that \tilde{p} is smooth too. Hence, for smooth a , both $\tilde{\rho}$ and \tilde{p} are non-singular.

The four-velocity vector field is $u = \frac{\partial}{\partial t}$, which is a smooth unit timelike vector. The densitized stress-energy tensor becomes therefore

$$T_{ab}\sqrt{-g} = (\tilde{\rho} + \tilde{p}) u_a u_b + \tilde{p} g_{ab}, \quad (6.17)$$

which is smooth, because $\tilde{\rho}$ and \tilde{p} are smooth functions. \square

Remark 6.2. We can write now a smooth densitized version of the Einstein Equation:

$$G_{ab}\sqrt{-g} + \Lambda g_{ab}\sqrt{-g} = \kappa T_{ab}\sqrt{-g}. \quad (6.18)$$

Remark 6.3. If $a(0) = 0$, the equation (6.15) tells us that $\tilde{\rho}(0) = 0$. From these and equation (6.16) we see that $\tilde{p}(0) = 0$ as well. Of course, this doesn't necessarily tell us that ρ or p are zero at $t = 0$, they may even be infinite. Figure 6.3 shows how the universe will look, in general.

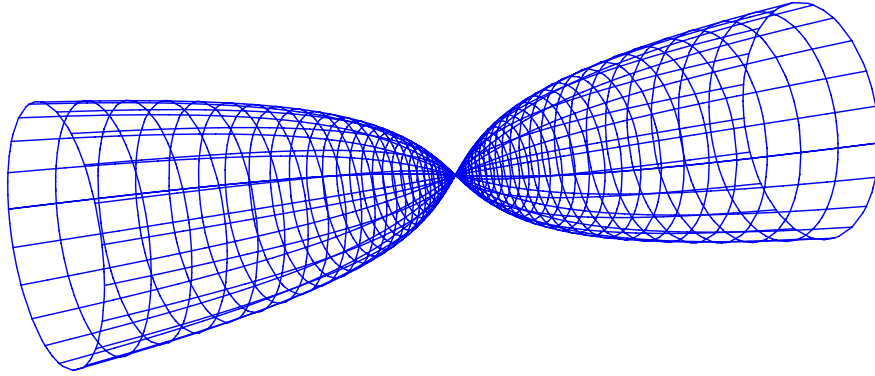


FIGURE 6.3: A schematic representation of a generic Big Bang singularity, corresponding to $a(0) = 0$. The universe can be continued before the Big Bang without problems.

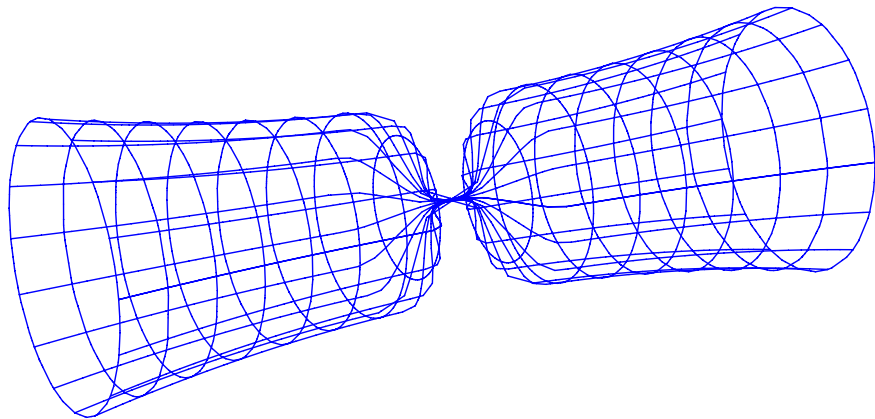


FIGURE 6.4: A schematic representation of a Big Bang similar to an infinitesimal Big Bounce, corresponding to $a(0) = 0$, $\dot{a}(0) = 0$, $\ddot{a}(0) > 0$.

Remark 6.4. One interesting possibility is that when $a(0) = 0$, also $\dot{a}(0) = 0$. In this case we may have $a(t) \geq 0$ around $t = 0$, for example if $\ddot{a}(0) > 0$, and obtain a Big Bang represented schematically in Fig. 6.4. This is very similar to a Big Bounce model, except that the singularity still appears.

Remark 6.5. Note that we preferred the stress-energy tensor with lowered indices, T_{ab} , to that with raised indices T^{ab} , because the former involves the smooth metric g_{ab} , while the latter involves its inverse, g^{ab} , which is singular when g_{ab} becomes degenerate. Similarly for the Einstein Equation. The two versions are equivalent only when the metric is non-degenerate.

6.1.4 The evolution of the universe

This chapter presented several scenarios concerning the Big Bang singularity, in the context of the Friedmann-Lemaître-Robertson-Walker model. It was found that the singularity is of degenerate type, and the time evolution is not obstructed. The solutions are schematically represented in Figures 6.3 and 6.4. These models only tell what happens at the singularity. At a global scale, the universe may re-collapse in a similar singularity and then pass again beyond it, in a cyclic cosmological model, or may expand accelerating forever, as the present day observations seem to suggest [114, 115]. Maybe the precedent universe, having $t < 0$, has no Big Bang at its origin, it just comes from the infinite past and collapses in a Big Crunch. Then, its Big Crunch becomes the Big Bang of our universe, and it starts its infinite expansion.

An alternative view, suggested by the time symmetry of the model, is that the pre-Big-Bang universe evolves backwards as compared to ours. This view is compatible with the idea that the entropy is lower at the Big-Bang. The cosmological arrow of our universe points from the Big Bang toward the time direction where the universe expands, which is the direction in which t increases ($t \rightarrow +\infty$). The cosmological arrow of the universe existing before the Big Bang points, of course, from the Big Bang, toward $-\infty$ (see Fig. 6.5).

It is often assumed that the entropic arrow of time is determined by some special conditions existing at the Big Bang. Of course, the “entropic arrow” may be undefined for a simple FLRW universe, but it may be defined in universes which are at large scale approximated by FLRW models. If the entropic arrow is determined by the cosmological arrow, then our model seems to suggest that the precedent universe has the entropic arrow of time oriented toward $-\infty$, and its time flows from the Big Bang toward $-\infty$, which is what the observers from the universe with $t > 0$ would call past.

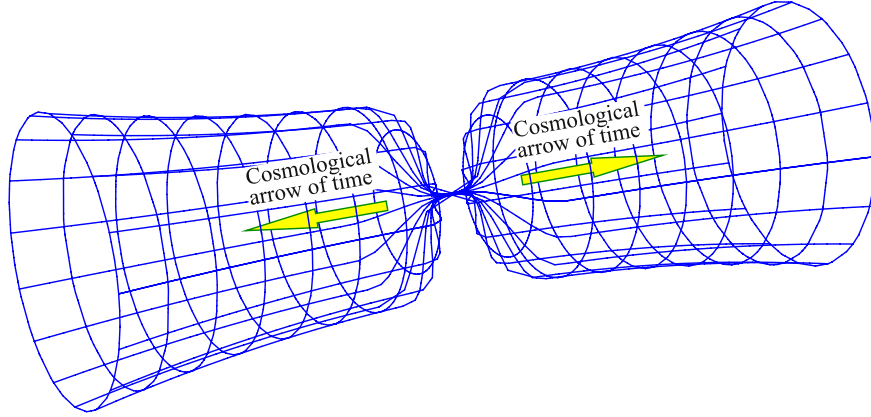


FIGURE 6.5: If the anterior universe has the cosmological arrow of time oriented toward our past, can we conclude that its entropic arrow of time also points toward our past?

6.2 FLRW Big-Bang singularity is quasi-regular

Einstein's equation, in its standard form, breaks down at the Big Bang singularity. We will see here that the expanded Einstein equation remains smooth at the Big Bang singularity of the Friedmann-Lemaître-Robertson-Walker model.

6.2.1 Introduction

Section §6.2.2 contains the central result, a theorem showing that the expanded Einstein equation is smooth everywhere, including at the Big Bang singularity. Section §6.2.3 discusses some properties of the proposed equation and solution. We conclude with some observations and implications in §6.2.4.

6.2.2 Beyond the FLRW Big-Bang singularity

Theorem 6.6. *For the FLRW metric (6.1), with $a : I \rightarrow \mathbb{R}$ a smooth function of time, the tensors R_{abcd} , S_{abcd} , and E_{abcd} are smooth, and consequently the expanded Einstein equation is smooth too, even when $a(t) = 0$.*

Proof. If we denote by $\tilde{T}_{ab} := \kappa T_{ab} - \Lambda g_{ab}$,

$$\begin{aligned}
 R_{ab} &= \tilde{T}_{ab} - \frac{1}{2} g^{st} \tilde{T}_{st} \\
 &= \kappa T_{ab} - \Lambda g_{ab} - \frac{\kappa}{2} g^{st} T_{st} g_{ab} + 2\Lambda g_{ab} \\
 &= \kappa \left(\rho + \frac{p}{c^2} \right) u_a u_b + \kappa p g_{ab} - \frac{\kappa}{2} (-\rho c^2 - p + 4p) g_{ab} + \Lambda g_{ab} \\
 &= \kappa \left(\rho + \frac{p}{c^2} \right) u_a u_b + \frac{\kappa}{2} (\rho c^2 - p) g_{ab} + \Lambda g_{ab}
 \end{aligned} \tag{6.19}$$

From equations (6.4), (6.5), and (6.19), we can see that the Ricci tensor has the form

$$R_{ab} = a^{-2}(t)\alpha(t)u_a u_b + a^{-2}(t)\beta(t)g_{ab}. \quad (6.20)$$

where $\alpha(t)$ and $\beta(t)$ are smooth functions. Similarly,

$$\begin{aligned} R &= g^{st}R_{st} \\ &= \kappa(-\rho c^2 - p + 2\rho c^2 - 2p) + 4\Lambda \\ &= \kappa(\rho c^2 - 3p) + 4\Lambda \end{aligned} \quad (6.21)$$

and there is a smooth function $\gamma(t)$ so that

$$R = a^{-2}(t)\gamma(t). \quad (6.22)$$

We need to check that $a^{-2}(t)$ is compensated in S_{abcd} and E_{abcd} , so that $a(t)$ appears to a non-negative power.

Since the FLRW metric (6.1) is diagonal in the standard coordinates, each term in $(g \circ g)_{abcd}$ is of the form $g_{aa}g_{bb}$, with $a \neq b$. This means that at least $a \neq t$ or $b \neq t$ holds, and from (6.1) we conclude that $g_{aa}g_{bb}$ contains $a(t)$ at least to the power 2. Therefore, the scalar part of the Riemann curvature, S_{abcd} , is smooth.

For the same reason, the Kulkarni-Nomizu product between the metric tensor and the term $a^{-2}(t)\beta(t)g_{ab}$ from the expression of the Ricci curvature (6.20) is smooth.

The only term from (6.20) we have to check that is smoothened by the Kulkarni-Nomizu product with g is $a^{-2}(t)u_a u_b$. Since $u_a = g_{as}u^s$ and g is diagonal, it follows that $u_a = g_{aa}u^a$ (without summation). If $b \neq t$ ($a \neq t$ is similar), then $u_b = g_{bb}u^b$ contains the needed $a^2(t)$. In the case when $a = b = t$, $a^{-2}(t)u_t u_t$ is not necessarily smooth, but in the Kulkarni-Nomizu product it will appear only in terms of the form $a^{-2}(t)u_t u_t g_{cc}$, with $c \neq t$. Hence, $\text{Ric} \circ g$ is smooth. From this and from the smoothness of S_{abcd} , it follows that E_{abcd} is also smooth.

One of the properties of the FLRW metric is that it is conformally flat, that is, $C_{abcd} = 0$. From this it follows that $R_{abcd} = S_{abcd} + E_{abcd}$ is smooth too. \square

In fact, from Theorem 5.10 this follows directly and more generally:

Corollary 6.7. *The FLRW spacetime, with smooth $a : I \rightarrow \mathbb{R}$, is quasi-regular.*

Proof. This is a direct consequence of Theorem 5.10. \square

Remark 6.8. The Corrolary 6.7 applies to any FLRW universe, not only those filled with a fluid. For this particular case, we gave a direct proof in [99], showing explicitly how the expected infinities of the physical fields cancel out.

6.2.3 Properties of the proposed equation

6.2.3.1 Conservation of energy

The conservation of energy is usually put in the form

$$-a^3\dot{\rho} = 3a^2\dot{a}\rho + \frac{3}{c^2}a^2\dot{a}p, \quad (6.23)$$

which remains valid even when the volume $a^3 \rightarrow 0$.

6.2.3.2 The metric is parallel

It is known that if the metric tensor is regular, its covariant derivative vanishes, $g_{ab;c} = 0$. For our solution, this is true so long as $a(t) \neq 0$. But if $a = 0$, the metric is degenerate, and we have to check that $g_{ab;c} = 0$.

The metric being diagonal, its Christoffel symbols of the first kind,

$$\Gamma_{abc} = \frac{1}{2}(g_{bc,a} + g_{ca,b} - g_{ab,c}) \quad (6.24)$$

which don't vanish are either of the form

$$\Gamma_{aaa} = \frac{1}{2}g_{aa,a} \quad (6.25)$$

or, for $a \neq b$,

$$\Gamma_{aab} = -\frac{1}{2}g_{aa,b} \quad (6.26)$$

or

$$\Gamma_{aba} = \Gamma_{baa} = \frac{1}{2}g_{aa,b} \quad (6.27)$$

Consequently, the Christoffel symbols of the second kind,

$$\Gamma_{ab}^c = g^{cs}\frac{1}{2}(g_{bs,a} + g_{sa,b} - g_{ab,s}) \quad (6.28)$$

are of the form

$$\Gamma_{aa}^a = \frac{1}{2}\frac{g_{aa,a}}{g_{aa}}(!) \quad (6.29)$$

or, for $a \neq b$,

$$\Gamma_{aa}^b = -\frac{1}{2} \frac{g_{aa,b}}{g_{bb}} (!) \quad (6.30)$$

or

$$\Gamma_{ab}^a = \Gamma_{ba}^a = \frac{1}{2} \frac{g_{aa,b}}{g_{aa}} (!) \quad (6.31)$$

where (!) means “no summation over the repeated indices”.

It follows that the covariant derivative introduces in the worst case a division by $a^2(t)$.

The covariant derivative of the metric tensor is

$$g_{ab;c} = g_{ab,c} - \Gamma_{bc}^s g_{as} - \Gamma_{ac}^s g_{sb}. \quad (6.32)$$

Obviously $g_{ab,c}$ is smooth, because g_{ab} is smooth. From the other terms, the only ones involving non-vanishing Christoffel symbols are of the form $\Gamma_{aa}^a g_{aa}$, $\Gamma_{bb}^a g_{aa}$, and $\Gamma_{ab}^a g_{aa}$, without summation. Whenever Γ_{bc}^a involves $a(t)$ to a negative power, which can only be 1 or 2, this is compensated by g_{aa} , which contains $a^2(t)$. It follows that the covariant derivative of the metric tensor is smooth, and by continuity is zero even when the metric becomes degenerate (at $a(t) = 0$):

$$g_{ab;c} = 0. \quad (6.33)$$

6.2.3.3 The Bianchi identity

We will show that the Riemann curvature tensor satisfies the Bianchi identity

$$R_{(abc)d;e} = 0. \quad (6.34)$$

Given that it holds at all the points for which $a(t) \neq 0$, where the metric is regular, it also holds by continuity at $a(t) = 0$. But we need to check that the covariant derivatives $(\text{Ric} \circ g)_{abcd;e}$ are smooth, because if the Bianchi identity would be between infinite values, there would be no continuity.

Since the Weyl part of the Riemann curvature $C_{abcd} = 0$ in the FLRW spacetime, and from the equations (6.20) and (6.22), it follows that R_{abcd} has the following form:

$$R_{abcd} = a^{-2}(t) \mu(t) ((u \otimes u) \circ g)_{abcd} + a^{-2}(t) \nu(t) (g \circ g)_{abcd}, \quad (6.35)$$

where the functions $\mu(t)$ and $\nu(t)$ are smooth.

Let's denote by h_{ij} , $1 \leq i, j \leq 3$, the metric on Σ . Then $g_{ij} = a^2(t) h_{ij}$. Given that our frame is comoving with the fluid, $u_t = 1$, and $u_i = 0$ for all i . The only

terms $a^{-2}(t)\mu(t)((u \otimes u) \circ g)_{abcd}$ which don't vanish by containing $u_i = 0$ are of the form $a^{-2}(t)\mu(t)u_t u_t g_{ii}$. The covariant derivatives with respect to t cancel one another in the Bianchi identity under the permutation, because the index t is repeated. So we check now those terms of the form $\nabla_j (a^{-2}(t)\mu(t)u_t u_t g_{ii})$, where $i \neq j$. But $\nabla_j (a^{-2}(t)\mu(t)u_t u_t g_{ii}) = a^{-2}(t)\mu(t)u_t u_t g_{ii;j} = 0$, because only g_{ii} depends on the space-like direction x^j , and because the metric tensor is parallel (6.33).

The terms $a^{-2}(t)\nu(t)(g \circ g)_{abcd}$ can only be of the form $a^{-2}(t)\nu(t)g_{aa}g_{bb}$, $a \neq b$. Since $\nabla_i (a^{-2}(t)\nu(t)g_{aa}g_{bb}) = a^{-2}(t)\nu(t)(g_{aa;i}g_{bb} + g_{aa}g_{bb;i}) = 0$, it follows that the covariant derivatives with respect to i vanish. We check now those with respect to t . If either the index a or b is equal to t , then the cyclic permutation involved in the Bianchi identity vanishes. Then the only remaining possibility is $\nabla_t (a^{-2}(t)\nu(t)g_{ii}g_{jj})$. But $\nabla_t (a^{-2}(t)\nu(t)g_{ii}g_{jj}) = -2\dot{a}(t)a^{-3}(t)\nu(t)g_{ii}g_{jj} + a^{-2}(t)\dot{\nu}(t)g_{ii}g_{jj} = -2\dot{a}(t)a(t)\nu(t)h_{ii}h_{jj} + a^2(t)\dot{\nu}(t)h_{ii}h_{jj}$, which is smooth.

Hence, the Bianchi identity makes sense even at the singularity $a(t) = 0$.

6.2.3.4 Action principle

Shortly after Einstein proposed his field equation, Hilbert and Einstein provided a Lagrangian formulation. The Lagrangian density which leads to Einstein's equation with matter given by $\mathcal{L}\sqrt{-g}$ and cosmological constant Λ is

$$\frac{1}{2\kappa} (R\sqrt{-g} - 2\Lambda\sqrt{-g}) + \mathcal{L}\sqrt{-g}. \quad (6.36)$$

In our case, the scalar curvature is singular at $a(t) \rightarrow 0$. But this doesn't affect the Lagrangian density, since the density $R\sqrt{-g}$ is smooth [98]. Given that our expanded Einstein equation (5.11) is equivalent to Einstein's its solutions are extremals of the action given by (6.36).

6.2.4 Conclusions

The new form of Einstein's equation extends uniquely beyond the Big Bang singularity, as it is represented schematically in Figure 6.6.

An alternative solution was proposed in section §5.1.2, where the Einstein equation was replaced with a densitized version (5.10).

The FLRW spacetime is an ideal one, based on the assumptions of the *cosmological principle* (that it is homogeneous and isotropic). But the extension proposed here opens new possibilities to explore.

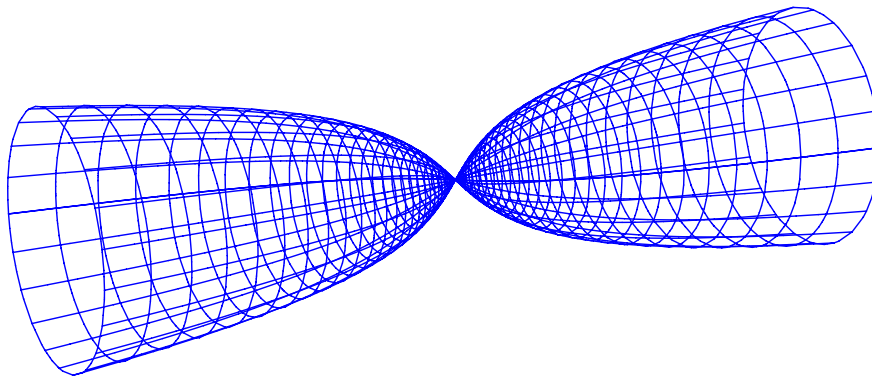


FIGURE 6.6: Schematic representation of a generic FLRW spacetime. The solutions of the new equation can be continued naturally before the Big Bang.

Singularities which are of the type studied here, having the Riemann curvature tensor R_{abcd} smooth, and admitting smooth Ricci decomposition, are in fact more general. In [101] the Schwarzschild singularity is put in a form in which has these properties, by an appropriate coordinate change. This, and similar results on the Reissner-Nordström singularity [96] suggests that we should reconsider the information loss [102]. More general cosmological models, which are neither homogeneous nor isotropic, are studied in [100], and shown to admit a smooth Ricci decomposition, and satisfy the Weyl curvature hypothesis [59]. Implications suggesting to reconsider the the problem of quantization are presented in [118, 119].

6.3 The Weyl curvature hypothesis

The Weyl curvature hypothesis of Penrose attempts to explain the high homogeneity and isotropy, and the very low entropy of the early universe, by conjecturing the vanishing of the Weyl tensor at the Big Bang singularity.

In previous chapters it has been proposed an equivalent form of Einstein's equation, which extends it and remains valid at an important class of singularities (including in particular the Schwarzschild, FLRW, and isotropic singularities). Here it is shown that if the Big Bang singularity is from this class, it also satisfies the Weyl curvature hypothesis.

As an application, we study a very general example of cosmological model, which generalizes the FLRW model by dropping the isotropy and homogeneity constraints. This model generalizes both the FLRW model and the isotropic singularities. We show that the Big-Bang singularity of this model is of the type under consideration, and satisfies therefore the Weyl curvature hypothesis.

6.3.1 Introduction

6.3.1.1 The Weyl curvature hypothesis

In searching an explanation of the second law of Thermodynamics, and of the high homogeneity and isotropy of the universe, especially around the Big Bang, Roger Penrose arrived at the *Weyl curvature hypothesis* (WCH) [59].

His physical motivation was the search for an explanation of the *arrow of time* – the second law, formulated as the *law of increasing entropy*. As it is known, Boltzmann’s explanation relies on the atomic structure of matter, in the framework of Newtonian mechanics. But the fundamental evolution equations in Newtonian mechanics are symmetric in time, in the sense that reversing the time in the equations leads to equally valid equations, with equally valid solutions. This explanation works also in the context of modern physics, because we don’t know of fundamental laws which are not invariant at time reversal (or at least at the simultaneous CPT transformation).

In terms of the coarse grained phase space, the entropy is proportional to the logarithm of the volume of the system. Systems at thermal equilibrium occupy the largest volume in phase space. Any state tends to evolve towards the most probable state, which is of highest entropy. The observations that entropy currently increases show that it was lower in the past, and that it was in fact more and more lower as one approaches the beginning of the Universe.

Penrose’s analysis lead him to the conclusion that the initial conditions of the Universe had to be restricted to a very small region of the phase space: of about $\frac{1}{10^{10^{123}}}$ (for comparison, the number of particles in the visible universe is of just 10^{90}). His analysis of the flow of energy in the Universe led him to the idea that the second law of Thermodynamics is due to very high homogeneity around the Big-Bang. Penrose explains this homogeneity by the following argument ([59], p. 614):

In terms of spacetime curvature, the absence of clumping corresponds, very roughly, to the absence of Weyl conformal curvature (since absence of clumping implies spatial-isotropy, and hence no gravitational principal null-directions).

He further added “this restriction on the early geometry should be something like: the Weyl curvature C_{abcd} vanishes at any initial singularity” ([59], p. 630).

The Weyl tensor C_{abcd} is the traceless part of the Riemann curvature tensor R_{abcd} . From gravitational viewpoint, it is responsible for the tidal forces. The tensor $C_{abc}{}^d$ is

invariant at conformal rescalings $g_{ab} \mapsto \Omega^2 g_{ab}$. If it vanishes, it indicates that the metric is conformally flat (in dimension ≥ 4 ; in lower dimension it vanishes trivially).

In addition to Penrose's motivations for the WCH, other reasons come from Quantum Gravity. We expect that, near the Big-Bang, the quantum effects of gravity take over, but we know that gravity is perturbatively nonrenormalizable at two loops [120, 121]. A vanishing Weyl tensor would mean a vanishing of local degrees of freedom, hence of gravitons, and this would remove some of the problems [122].

An important type of singularities which automatically satisfy a version of the WCH (stating that the Weyl curvature remains finite at singularity) are the *isotropic singularities*. They were researched by Tod [87–92], Claudel & Newman [93], Anguige & Tod [94, 95]. The metric tensor in this case can be obtained by a conformal rescaling from a regular (*i.e.* non-singular) metric tensor, and presents nice behavior from conformal geometric viewpoint. Because the Weyl tensor $C_{abc}{}^d$ is invariant at conformal rescalings, the main feature of the isotropic singularities is that the Weyl tensor equals that of a regular metric, hence remaining finite. If we apply a conformal rescaling to a metric tensor g_{ab} , the new metric $\hat{g}_{ab} := \Omega^2 g_{ab}$ has the Weyl curvature tensor $\hat{C}_{abc}{}^d = C_{abc}{}^d$, which is smooth, but not necessarily vanishing at $\Omega = 0$. But \hat{C}_{abcd} vanishes, since

$$\hat{C}_{abcd} = \hat{g}_{sd} \hat{C}_{abc}{}^s = \Omega^2 g_{sd} C_{abc}{}^s = 0. \quad (6.37)$$

A simple example of vanishing Weyl tensor is provided by the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model. This model implements the *cosmological principle* that the Universe is, at very large scales, homogeneous and isotropic. But it is irrelevant for the WCH, because it is too symmetric, and its Weyl tensor vanishes identically.

In this chapter we will prove that a much larger class of singularities, which includes among others the isotropic singularities, satisfies the Weyl curvature hypothesis. This larger class is not a random generalization of the isotropic singularities, but it appeared from our research on a different problem. Previous considerations led us to the idea that, for a singularity to be still manageable from mathematical viewpoint, some conditions are in order [67, 73]. These conditions allowed the definition of the Riemann curvature R_{abcd} (but not of $R_{abc}{}^d$). This program successfully led to a better understanding of the black hole singularities (see *e.g.* [96]).

Further considerations suggested that the Ricci decomposition of the curvature tensor should be smooth, in order to allow the writing of an equation which extends Einstein's at a large class of singularities, but is equivalent to it for non-degenerate metrics [84]. These singularities behave, mathematically, as if the metric loses one or more dimensions,

and consequently as if the cotangent space loses one or more dimensions. They have, as a bonus, the property that C_{abcd} vanishes, as we will prove in §6.3.2. The proof is largely based on the idea that $C_{abcd} = 0$ in lower dimensions.

As an application, in §6.3.3 we will study a very general cosmological model, which does not assume, like the FLRW model, that the space slices have the metric constant in time, up to an overall scaling factor $a(t)$ which depends on the time only and which vanishes at the Big-Bang. In our model we will keep the overall scaling factor $a(t)$, but we will allow the space part of the metric to change in time freely. The physical motivation is that in the actual Universe there is no perfect isotropy and homogeneity, and in fact at smaller scales the inhomogeneity is important. Working in such general settings, we will not be concerned at this time with the matter content of this universe. Because the solution is very general, the possibilities of matter fields which can give this kind of metric are limitless. This metric has non-vanishing Weyl tensor in general, but we will show that at the Big-Bang singularities C_{abcd} vanishes.

6.3.2 The Weyl curvature vanishes at quasi-regular singularities

Theorem 6.9. *The Weyl curvature tensor C_{abcd} vanishes at quasi-regular singularities.*

Proof. From the smoothness of R_{abcd} , E_{abcd} , S_{abcd} , and from equation (5.17), follows the smoothness of C_{abcd} .

The following considerations use objects described in detail in [67] (here in chapter 2), but we try to make the proof as self-contained as possible. Since the metric g is a bilinear form on the tangent vector space $T_p M$, it defines, at the points p where it is degenerate, the totally degenerate space $T_{\circ p} M := T_p M^\perp$, named the *radical* of $T_p M$. We construct its annihilator

$$T^\bullet_p M := \{ \omega \in T_p^* M; \omega|_{T_{\circ p} M} = 0 \}, \quad (6.38)$$

named the *radical annihilator*. The radical annihilator is the image of the index lowering morphism $\flat : T_p M \rightarrow T_p^* M$, $X^\flat(Y) := \langle X, Y \rangle$, $\forall X, Y \in T_p M$:

$$T^\bullet_p M = \text{im } \flat \leq T_p^* M. \quad (6.39)$$

The dual of the radical annihilator, $(T^\bullet_p M)^*$, is isomorphic with the quotient $T_p M / T_{\circ p} M$. On $T^\bullet_p M$, g induces a canonical non-degenerate metric defined by

$$g_\bullet(\omega, \tau) := \langle X, Y \rangle \quad (6.40)$$

where $X^\flat = \omega$ and $Y^\flat = \tau$. This metric is used for covariant contractions, so long as the contracted tensor components live in $T^\bullet_p M$.

As it is shown in [67], the Riemann curvature tensor R_{abcd} of a semi-regular semi-Riemannian manifold satisfies at each $p \in M$

$$(R_{abcd})_p \in T^\bullet_p M \otimes T^\bullet_p M \otimes T^\bullet_p M \otimes T^\bullet_p M. \quad (6.41)$$

At the points p where the metric g is degenerate, $\dim(T^\bullet_p M) < 4$. Since in dimension ≤ 3 any tensor having the symmetries of the Weyl tensor vanishes (see *e.g.* [86]), it follows that

$$(C_{abcd})_p = 0 \quad (6.42)$$

whenever g is degenerate at p . This concludes the proof. \square

6.3.3 Example: a general, non-isotropic and inhomogeneous cosmological model

The FLRW model is very good for very large scales, where we can completely ignore any inhomogeneity and anisotropy in the distribution of matter. But in reality the universe is not homogeneous at all scales. This is why we are motivated in studying a spacetime (M, g) which is allowed to be inhomogeneous and anisotropic. We assume that there is a global time $\tau : M \rightarrow I$ where $I \subseteq \mathbb{R}$ is an interval. We also assume that the topology of the space slices $\Sigma_t := \tau^{-1}(t)$ is independent on the time $t \in I$, being thus all of them diffeomorphic with a three-dimensional manifold Σ . Each manifold Σ_t is endowed with a metric tensor of the form

$$g_{ij}(t) := a(t)h_{ij}(t), \quad (6.43)$$

where $1 \leq i, j \leq 3$, $h(t)$ is a Riemannian (non-degenerate) metric on Σ_t , also represented as arc element by $d\sigma_t$. We require $a(t)$ and $h_{ij}(t)$ to depend smoothly on $t \in I$. While $h(t)$ is Riemannian (and non-degenerate) for any $t \in I$, $a(t)$ is allowed to vanish. The Big-Bang is therefore obtained for $a(t) = 0$.

Therefore we see that it is natural to consider that the spacetime is $M = I \times \Sigma$, with the following metric:

$$ds^2 = -dt^2 + a^2(t)d\sigma_t^2. \quad (6.44)$$

If we take $h_{ij}(t)$ to be independent on time, and of constant curvature, we obtain the FLRW model (which is not interesting from this viewpoint, because in this case $C_{abcd} = 0$ trivially, for all times t). But if we allow $d\sigma_t^2$ to depend freely on time, the solution is much more general. The Big Bang singularity obtained when $a(t) = 0$ is much more

general than the FLRW Big Bang singularity, because we allow the geometry of space slices $(\Sigma_t, h(t))$ to be inhomogeneous and to vary with time.

We will actually prefer to be a bit more general and allow the metric on I to become degenerate too:

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\sigma_t^2, \quad (6.45)$$

where $N : I \rightarrow \mathbb{R}$ is a smooth function. If $N(t) \neq 0$ for any $t \in I$, then a reparametrization of I allows the metric to be of the form (6.44), so this generalization is important only when $N(t)$ vanishes together with $a(t)$. For reasons which will become apparent, we will require that

$$f(t) := \frac{a(t)}{N(t)} \quad (6.46)$$

is not singular. For example, if $f(t) = 1$ (hence $N(t) = a(t)$), the resulting singularities are just isotropic singularities, as those studied by Tod & *al.*. But allowing $f(t)$ to vanish together with $a(t)$ leads to more general, anisotropic singularities.

We are here interested in the most general case.

Theorem 6.10. *The metric (6.45) is quasi-regular.*

Proof. The plan is to prove that the metric is semi-regular, by showing that the terms in the Riemann curvature tensor (2.81) are smooth. Then we show that the Ricci decomposition

$$R_{abcd} = E_{abcd} + S_{abcd} + C_{abcd}. \quad (6.47)$$

is smooth.

The metric components are:

$$g(t, x) = \begin{pmatrix} g_{00} & g_{0i} \\ g_{j0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2(t) & 0 \\ 0 & a^2(t)h_{ij}(t, x) \end{pmatrix} \quad (6.48)$$

The reciprocal metric components are:

$$g^{-1}(t, x) = \begin{pmatrix} g^{00} & g^{0i} \\ g^{j0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -N^{-2}(t) & 0 \\ 0 & a^{-2}(t)h^{ij}(t, x) \end{pmatrix} \quad (6.49)$$

The partial derivatives of the metric tensor are therefore:

$$\begin{aligned}
g_{00,0} &= -2N\dot{N} \\
g_{00,k} &= 0 \\
g_{ij,0} &= a(2\dot{a}h_{ij} + a\dot{h}_{ij}) \\
g_{ij,k} &= a^2\partial_k h_{ij}
\end{aligned} \tag{6.50}$$

The second order partial derivatives of the metric tensor are:

$$\begin{aligned}
g_{00,00} &= -2(\dot{N}^2 + N\ddot{N}) \\
g_{00,k0} &= g_{00,0k} = g_{00,kl} = 0 \\
g_{ij,00} &= 2\dot{a}^2 h_{ij} + 2a\ddot{a}h_{ij} + 4a\dot{a}\dot{h}_{ij} + a^2\ddot{h}_{ij} \\
g_{ij,k0} &= a(2\dot{a}\partial_k h_{ij} + a\partial_k \dot{h}_{ij}) \\
g_{ij,kl} &= a^2\partial_k \partial_l h_{ij}
\end{aligned} \tag{6.51}$$

To check that g is semi-regular, it is enough to check that the terms of the form $g_{ab,\bullet}g_{cd,\bullet}$ are smooth. By using the equations (6.50) we find that

$$\begin{aligned}
g_{00,\bullet}g_{00,\bullet} &= -N^{-2}g_{00,0}g_{00,0} + a^{-2}h^{cd}g_{00,c}g_{00,d} \\
&= -4\dot{N}^2
\end{aligned} \tag{6.52}$$

$$\begin{aligned}
g_{00,\bullet}g_{ij,\bullet} &= -N^{-2}g_{00,0}g_{ij,0} + a^{-2}h^{cd}g_{00,c}g_{ij,d} \\
&= 2\frac{\dot{N}a}{N}(2\dot{a}h_{ij} + a\dot{h}_{ij})
\end{aligned} \tag{6.53}$$

$$\begin{aligned}
g_{ij,\bullet}g_{kl,\bullet} &= -N^{-2}g_{ij,0}g_{kl,0} + a^{-2}h^{cd}g_{ij,c}g_{kl,d} \\
&= -\frac{a^2}{N^2}(2\dot{a}h_{ij} + a\dot{h}_{ij})(2\dot{a}h_{kl} + a\dot{h}_{kl}) + a^2h^{cd}\partial_c h_{ij}\partial_d h_{kl}
\end{aligned} \tag{6.54}$$

But since for a smooth function $f(t, x)$

$$a(t, x) = f(t, x)N(t), \tag{6.55}$$

the terms calculated above become now manifestly smooth:

$$\begin{aligned}
g_{00,\bullet}g_{00,\bullet} &= -4\dot{N}^2 \\
g_{00,\bullet}g_{ij,\bullet} &= 2\dot{N}f(2\dot{a}h_{ij} + a\dot{h}_{ij}) \\
g_{ij,\bullet}g_{kl,\bullet} &= -f^2(2\dot{a}h_{ij} + a\dot{h}_{ij})(2\dot{a}h_{kl} + a\dot{h}_{kl}) + a^2h^{cd}\partial_c h_{ij}\partial_d h_{kl}
\end{aligned} \tag{6.56}$$

Therefore the metric g is semi-regular.

We are now interested in proving that $\text{Ric} \circ g$ and $Rg \circ g$ are smooth. For this, we have to contract the terms from (6.51) and (6.56), and see what happens when taking Kulkarni-Nomizu products with g . If we apply a contraction, to calculate the Ricci tensor, each of these terms will get an additional factor of N^{-2} or $a^{-2} = f^{-2}N^{-2}$, which we hope will be canceled by the terms N^2 and a^2 introduced by the Kulkarni-Nomizu products with g .

So let's analyze each of the terms from (6.51) and (6.56). We are not interested in the terms $g_{00,k0}$, $g_{00,0k}$, and $g_{00,kl}$, because they vanish.

The term $g_{00,00}$ can be contracted only by g^{00} . One such contraction, needed to get the Ricci tensor, introduces the factor N^{-2} , which is canceled by taking the Kulkarni-Nomizu product with the metric g , which introduces the counterfactor N^2 (since it introduces $a^2 = f^2N^2$). Similarly, two contractions, needed to obtain the curvature scalar, introduce the factor N^{-4} , canceled by the factor N^4 contained in $g \circ g$. Similar reasoning leads to the conclusion that the terms obtained from $g_{00,\bullet}g_{00,\bullet}$ are smooth too.

For the terms of the form $g_{ij,kl}$, or those of the form $g_{ij,\bullet}g_{kl,\bullet}$, we have the following situation. Luckily, $g_{ij,kl}$ already contains the factor a^2 , while

$$g_{ij,\bullet}g_{kl,\bullet} = f^2 \left(-(2\dot{a}h_{ij} + a\dot{h}_{ij})(2\dot{a}h_{kl} + a\dot{h}_{kl}) + N^2 h^{cd} \partial_c h_{ij} \partial_d h_{kl} \right) \quad (6.57)$$

already contains the factor f^2 . A first contraction will introduce $a^{-2} = f^{-2}N^{-2}$, but f^2 was already present in the initial term. Multiplying with g_{00} introduces the counterfactor N^2 , which is enough to cancel the remaining factor N^{-2} . Noticing that $g \circ g$ contains g_{00} at most once in a given term, we conclude that the factors introduced by the second contraction are also canceled by taking the Kulkarni-Nomizu product.

When contracting the terms of the form $g_{ij,00}$, the contractions introduce the factors $a^{-2}N^{-2}$. Since a term of $g \circ g$ contains at most one factor g_{00} , this ensures the overall cancelation of $a^{-2}N^{-2}$. The worst case is in calculating E_{abcd} , when the first contraction is with g^{ij} , because this introduces the term a^{-2} first. But then the Kulkarni-Nomizu product between resulting contracted term $g^{ij}g_{ij,00}$ and g will have only terms of the form $g^{ij}g_{ij,00}g_{kl}$ (the terms $g^{ij}g_{ij,00}g_{00}$ being canceled by taking the symmetries in (5.12)), and since $g_{kl} = a^2h_{kl}$, it cancels the factor a^{-2} . The terms of the form $g_{00,\bullet}g_{ij,\bullet}$ behave similarly.

It remains to see what happens with the terms obtained from the terms of the form $g_{ij,k0} = a \left(2\dot{a}\partial_k h_{ij} + a\partial_k \dot{h}_{ij} \right)$. The term $a^2\partial_k \dot{h}_{ij}$ contains a^2 , so it doesn't pose problems. The only interesting term is $A_{ijk0} := a\dot{a}\partial_k h_{ij}$ (or other permutations of the i, j, k indices). A contraction with g^{ij} introduces a^{-2} , and we obtain a term of the form $a^{-1}A_{k0}$. The

Kulkarni-Nomizu product of A_{k0} with g has the form

$$A_{0j}g_{ki} - A_{0i}g_{kj}, \quad (6.58)$$

because $g_{0i} = g_{0j} = 0$. We know that a Riemannian metric in a space of dimension ≤ 3 can always be diagonalized [123], so if we take $h_{ij}(t)$ to be diagonal at the t in which we are interested, the above terms cancel one another.

This concludes the proof that all terms contained in R_{abcd} , E_{abcd} , and S_{abcd} are smooth. Hence, the metric (6.45) is quasi-regular. \square

This model can be connected with a cosmological model proposed by P. Fiziev and D. V. Shirkov [124], which consists in a dimensional reduction of spacetime (in topology and geometry) at the Big-Bang. When $a(t)$ vanishes, space can be considered, in a way, to shrink to a point, so that one remains only with the time dimension. There are several properties of our model which seem to support that, at least, the universe behaves “as if” the dimensionality is reduced: the reduction of the rank of the metric, of the dimension of the cotangent space at such singular points, and by a certain independence of fields with the degenerate directions. In fact, if $N(t) = 0$ as well, it appears that the Universe emerged from a dimensionless point. Yet, the conditions in which such a dimensional reduction can be considered are not clear. Moreover, for our quasi-regular singularities it seems to be, at least at this time, better to maintain four topological dimensions, because such a reduction may, in general, lose initial data.

6.3.4 Conclusion

The semi-regular singularities are well-behaved from many viewpoints, allowing us to perform the most important operations which are allowed by semi-Riemannian manifolds with regular metric tensor. When they are in addition quasi-regular, we can write a smooth expanded version of Einstein’s equation (§5.2.2).

The quasi-regular singularities offer a nice surprise, since they have vanishing Weyl curvature C_{abcd} . It follows that any quasi-regular Big Bang singularity also satisfies the Weyl curvature hypothesis (§6.3.2).

As a main application, we studied in §6.3.3 a cosmological model which extends FLRW, by dropping the isotropy and homogeneity conditions. This generality is more realistic from physical viewpoint, since our Universe appears homogeneous and isotropic only at very large scales. This model contains as particular cases, in addition to FLRW, also the isotropic singularities.

Chapter 7

The black hole singularities

In this chapter, author's papers [96, 97, 101, 102] have been used.

7.1 Schwarzschild singularity is semi-regularizable

It is shown that the Schwarzschild spacetime can be extended so that the metric becomes analytic at the singularity. The singularity continues to exist, but it is made degenerate and smooth, and the infinities are removed by an appropriate choice of coordinates. A family of analytic extensions is found, and one of these extensions is semi-regular. A degenerate singularity doesn't destroy the topology, and when is semi-regular, it allows the field equations to be rewritten in a form which avoids the infinities, as it was shown elsewhere [67, 73]. In the new coordinates, the Schwarzschild solution extends beyond the singularity. This suggests a possibility that the information is not destroyed in the singularity, and can be restored after the evaporation.

7.1.1 Introduction

The Schwarzschild black hole solution, expressed in the Schwarzschild coordinates, has the following metric tensor:

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2, \quad (7.1)$$

where

$$d\sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2 \quad (7.2)$$

is the metric of the unit sphere S^2 , m the mass of the body, and the units were chosen so that $c = 1$ and $G = 1$ (see *e.g.* [5], p. 149).

The first two terms in the right hand side of equation (7.1) don't depend on the coordinates θ and ϕ , and $d\sigma^2$ is independent on the coordinates r and t . Therefore we can view this solution as a warped product between a two-dimensional semi-Riemannian space and the sphere S^2 with the canonical metric (7.2). We can use this property to change the coordinates r and t independently, ignoring in calculations the term $r^2 d\sigma^2$, which we reintroduce at the end.

The singularity at $r = 2m$, which makes the coefficient $\left(1 - \frac{2m}{r}\right)^{-1}$ become infinite, is only apparent, as shown by the Eddington-Finkelstein coordinates ([65, 66], [5], p. 150).

As $r \searrow 0$, the coefficient $\left(1 - \frac{2m}{r}\right)^{-1}$ tends to 0, and the coefficient $-\left(1 - \frac{2m}{r}\right)$ tends to $+\infty$. This is a genuine singularity, as we can see from the fact that the scalar $R_{abcd}R^{abcd}$ tends to ∞ . This seems to suggest that the Schwarzschild metric cannot be made smooth at $r = 0$. In fact, as we will see, we can find coordinate systems in which the components of the metric, although degenerate, are analytic (hence they are finite), even at the genuine singularity given by $r = 0$. Moreover, we will see that we can find an analytic extension of the Schwarzschild spacetime, which is *semi-regular*.

In [67, 73, 76] it was developed the *singular semi-Riemannian geometry* for metrics which are allowed to change their signature, in particular to be degenerate. Such metrics g_{ab} are smooth, but g^{ab} tends to ∞ when the metric becomes degenerate. The notion of Levi-Civita connection cannot be defined, and the curvature cannot be defined canonically. But in the special case of semi-regular metrics we can construct a canonical Riemann curvature tensor R_{abcd} , which is smooth, although $R^a{}_{bcd}$ is not canonically defined and is singular. It admits canonical Ricci and scalar curvatures, which may be discontinuous or infinite at the points where the metric changes its signature. The usual tensorial and differential operations, normally obstructed by the degeneracy of the metric, can be replaced by equivalent operations which work fine, if the metric is semi-regular.

In this chapter we will show that the Schwarzschild solution can be extended analytically to such a well behaved semi-regular solution.

7.1.2 Analytic extension of the Schwarzschild spacetime

Theorem 7.1. *The Schwarzschild metric admits an analytic extension at $r = 0$.*

Proof. It is enough to make the coordinate change in a neighborhood of the singularity – in the region $r < 2m$. On that region, the coordinate r is timelike, and t is spacelike.

Let's change the coordinates by

$$\begin{cases} t &= t(\xi, \tau) \\ r &= r(\xi, \tau) \end{cases} \quad (7.3)$$

Recall that the metric coefficients in the Schwarzschild coordinates are

$$g_{tt} = \frac{2m - r}{r}, \quad g_{rr} = -\frac{r}{2m - r}, \quad g_{tr} = g_{rt} = 0. \quad (7.4)$$

In the new coordinates, the metric coefficients are

$$g_{\tau\tau} = \left(\frac{\partial r}{\partial \tau} \right)^2 g_{rr} + \left(\frac{\partial t}{\partial \tau} \right)^2 g_{tt} \quad (7.5)$$

$$g_{\tau\xi} = \frac{\partial r}{\partial \tau} \frac{\partial r}{\partial \xi} g_{rr} + \frac{\partial t}{\partial \tau} \frac{\partial t}{\partial \xi} g_{tt} \quad (7.6)$$

$$g_{\xi\xi} = \left(\frac{\partial r}{\partial \xi} \right)^2 g_{rr} + \left(\frac{\partial t}{\partial \xi} \right)^2 g_{tt} \quad (7.7)$$

From (7.4) we see that g_{rr} is analytic around $r = 0$, hence the only condition for the partial derivatives of r with respect to τ and ξ is that they are smooth. The expression of g_{tt} on the other hand, has r as denominator, hence we have to cancel it. From equations (7.5–7.7), we see that r as the denominator in the expression of g_{tt} is canceled if the partial derivatives of t have the form:

$$\begin{cases} \partial t / \partial \tau &= \rho F_\tau \\ \partial t / \partial \xi &= \rho F_\xi \end{cases} \quad (7.8)$$

where ρ , F_τ and F_ξ are smooth functions in τ and ξ , and

$$r = \rho^2(\tau, \xi). \quad (7.9)$$

The conditions (7.8) are satisfied for example if t has the form $t = \xi \rho^2$.

The metric becomes

$$g_{\tau\tau} = -\frac{4\rho^4}{2m - \rho^2} \left(\frac{\partial \rho}{\partial \tau} \right)^2 + (2m - \rho^2) F_\tau^2 \quad (7.10)$$

$$g_{\tau\xi} = -\frac{4\rho^4}{2m - \rho^2} \frac{\partial \rho}{\partial \tau} \frac{\partial \rho}{\partial \xi} + (2m - \rho^2) F_\tau F_\xi \quad (7.11)$$

$$g_{\xi\xi} = -\frac{4\rho^4}{2m - \rho^2} \left(\frac{\partial \rho}{\partial \xi} \right)^2 + (2m - \rho^2) F_\xi^2 \quad (7.12)$$

We can see now that the new metric is smooth around $r = 0$ – none of its components become infinite at $r = 0$. It is singular, because it is degenerate, but it is smooth. If the functions ρ , F_τ and F_ξ are analytic, so is the new metric.

To go back to the four-dimensional spacetime, we take the warped product between the above metric and the sphere S^2 , with the warping function ρ^2 . The warping function ρ^2 is smooth and cancels at the singular points $r = 0$. Hence, according to the central theorem of degenerate warped products from [73], the warped product between the two-dimensional extension (τ, ξ) and the sphere S^2 , with warping function ρ^2 , is degenerate. This is the needed extension of the Schwarzschild solution. \square

Corollary 7.2. *The metric in the coordinates $(\tau, \xi, \theta, \phi)$ is*

$$ds^2 = -\frac{4\rho^4}{2m - \rho^2}d\rho^2 + (2m - \rho^2)(F_\tau d\tau + F_\xi d\xi)^2 + \rho^4 d\sigma^2 \quad (7.13)$$

Proof. From (7.8) it follows that

$$dt = \rho(F_\tau d\tau + F_\xi d\xi). \quad (7.14)$$

From (7.9) it follows that

$$dr = 2\rho d\rho. \quad (7.15)$$

By substituting t and r into (7.1) we get the result. \square

Corollary 7.3. *The determinant of the metric in the new coordinates is*

$$\det g = -4\rho^4 \left(F_\tau \frac{\partial \rho}{\partial \xi} - F_\xi \frac{\partial \rho}{\partial \tau} \right)^2 \quad (7.16)$$

Proof. Direct calculation gives

$$\det g = g_{tt}g_{rr} \begin{vmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial \xi} \\ \frac{\partial r}{\partial \tau} & \frac{\partial r}{\partial \xi} \end{vmatrix}^2 = - \begin{vmatrix} \rho F_\tau & \rho F_\xi \\ 2\rho \frac{\partial \rho}{\partial \tau} & 2\rho \frac{\partial \rho}{\partial \xi} \end{vmatrix}^2 = -4\rho^4 \begin{vmatrix} F_\tau & F_\xi \\ \frac{\partial \rho}{\partial \tau} & \frac{\partial \rho}{\partial \xi} \end{vmatrix}^2$$

\square

Remark 7.4. The common belief is that it is impossible to extend the Schwarzschild metric so that it becomes smooth at the singularity, instead of becoming infinite. But this can be done, if we understand that the Schwarzschild coordinates are singular at $r = 0$ (and so are the other known coordinate systems for the Schwarzschild metric). To pass to a regular coordinate system from a singular one, we need to use a coordinate change which is singular, and the singularity in the coordinate change coincides with the

singularity of the metric. This can be viewed as analogous to the Eddington-Finkelstein coordinate transformation, which removes the apparent singularity on the event horizon [65, 66]. In both cases the metric is made smooth at points where it was thought to be infinite, the only difference is that in our case, at $r = 0$, the metric becomes degenerate.

Example 7.1. *The function ρ has the simplest expression when it depends on only one of the two variables, say τ . To find a coordinate change as we want, let's assume that ρ is the simplest function of τ , $\rho = \tau$.*

The metric becomes

$$g_{\tau\tau} = -\frac{4\tau^4}{2m - \tau^2} + (2m - \tau^2)F_\tau^2 \quad (7.17)$$

$$g_{\tau\xi} = (2m - \tau^2)F_\tau F_\xi \quad (7.18)$$

$$g_{\xi\xi} = (2m - \tau^2)F_\xi^2 \quad (7.19)$$

The determinant of the metric becomes

$$\det g = -4\tau^4 F_\xi^2. \quad (7.20)$$

The four-metric takes the form

$$ds^2 = -\frac{4\tau^4}{2m - \tau^2}d\tau^2 + (2m - \tau^2)(F_\tau d\tau + F_\xi d\xi)^2 + \tau^4 d\sigma^2 \quad (7.21)$$

Example 7.2. *The Example (7.1) simplified the form of $r(\tau, \xi)$. We can, in addition, simplify $t(\tau, \xi)$. Equation (7.8) suggests that we take t is a product between a power of τ and a function of τ and ξ . Let's assume that it has the form $\xi\tau^T$, where $T \geq 2$ in order to satisfy (7.8). Hence,*

$$\begin{cases} r &= \tau^2 \\ t &= \xi\tau^T \end{cases} \quad (7.22)$$

Then, we have

$$\frac{\partial r}{\partial \tau} = 2\tau, \quad \frac{\partial r}{\partial \xi} = 0, \quad \frac{\partial t}{\partial \tau} = T\xi\tau^{T-1}, \quad \frac{\partial t}{\partial \xi} = \tau^T \quad (7.23)$$

and

$$\begin{cases} F_\tau &= T\xi\tau^{T-2} \\ F_\xi &= \tau^{T-1} \end{cases} \quad (7.24)$$

The metric takes the form

$$g_{\tau\tau} = -\frac{4\tau^4}{2m - \tau^2} + T^2\xi^2(2m - \tau^2)\tau^{2T-4} \quad (7.25)$$

$$g_{\tau\xi} = T\xi(2m - \tau^2)\tau^{2T-3} \quad (7.26)$$

$$g_{\xi\xi} = (2m - \tau^2)\tau^{2T-2} \quad (7.27)$$

and its determinant

$$\det g = -4\tau^{2T+2}. \quad (7.28)$$

The four-metric is

$$ds^2 = -\frac{4\tau^4}{2m - \tau^2} d\tau^2 + (2m - \tau^2)\tau^{2T-4} (T\xi d\tau + \tau d\xi)^2 + \tau^4 d\sigma^2 \quad (7.29)$$

Remark 7.5. When we pass from one coordinate system (τ, ξ) characterized by T to another (τ', ξ') , characterized by $T' \neq T$, as in (7.22), the transformation has the Jacobian singular at $r = 0$. To check this, we use $r = \tau^2 = \tau'^2$. The Jacobian is then

$$J = \begin{pmatrix} \frac{\partial \tau}{\partial \tau'} & \frac{\partial \tau}{\partial \xi'} \\ \frac{\partial \xi}{\partial \tau'} & \frac{\partial \xi}{\partial \xi'} \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \tau^{T'-T} \end{pmatrix}, \quad (7.30)$$

and it is singular at $\tau = 0$. This seems to suggest that the different coordinate systems we found in the Example 7.2 represent distinct solutions. This raises the following open question.

Open Problem 7.6. Can we find natural conditions ensuring the uniqueness of the analytic extensions of the Schwarzschild solution at the singularity $\tau = 0$? Or can we consider all analytic extensions of this type to be equivalent, via coordinate changes which may be singular?

To support the second possibility, we can make the observation that the Jacobian from the Remark 7.5 is degenerate when we pass from a coordinate system characterized by $T = 2$ to another one with another value of T , but the converse is not true.

7.1.3 Semi-regular extension of the Schwarzschild spacetime

In section §7.1.2 we found an infinite family of coordinate changes which make the metric smooth. As we shall see now, among these solutions there is one which ensures the semi-regularity of the metric.

Theorem 7.7. *The Schwarzschild metric admits an analytic extension in which the singularity at $r = 0$ is semi-regular.*

Proof. To show that the metric is semi-regular, it is enough to show that there is a coordinate system in which the products of the form

$$g^{st} \Gamma_{abs} \Gamma_{cdt} \quad (7.31)$$

are all smooth [67], where Γ_{abc} are Christoffel's symbols of the first kind. In a coordinate system in which the metric is smooth, as in §7.1.2, Christoffel's symbols of the first kind are also smooth. But the inverse metric g^{st} is not smooth for $r = 0$. We will show that the products from the expression (7.31) are smooth.

We use the solution from the Example 7.2, and try to find a value for T , so that the metric is semi-regular.

The inverse of the metric has the coefficients given by $g^{\tau\tau} = g_{\xi\xi}/\det g$, $g^{\xi\xi} = g_{\tau\tau}/\det g$, and $g^{\tau\xi} = g^{\xi\tau} = -g_{\tau\xi}/\det g$. It follows from (7.25–7.27) that

$$g^{\tau\tau} = -\frac{1}{4}(2m - \tau^2)\tau^{-4} \quad (7.32)$$

$$g^{\tau\xi} = \frac{1}{4}T\xi(2m - \tau^2)\tau^{-5} \quad (7.33)$$

$$g^{\xi\xi} = \frac{\tau^{-2T+2}}{2m - \tau^2} - \frac{1}{4}T^2\xi^2(2m - \tau^2)\tau^{-6} \quad (7.34)$$

Christoffel's symbols of the first kind are given by

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}), \quad (7.35)$$

so we have to calculate the partial derivatives of the coefficients of the metric.

From (7.23) and (7.25–7.27) we have:

$$\begin{aligned} \partial_\tau g_{\tau\tau} &= \partial_\tau \left(-\frac{4\tau^4}{2m - \tau^2} + \xi^2 T^2 (2m - \tau^2) \tau^{2T-4} \right) \\ &= -4 \frac{4\tau^3(2m - \tau^2) + 2\tau^5}{(2m - \tau^2)^2} + 2T^2(2T - 4)m\xi^2 \tau^{2T-5} \\ &\quad - T^2(2T - 2)\xi^2 \tau^{2T-3}, \end{aligned}$$

hence

$$\partial_\tau g_{\tau\tau} = 8 \frac{\tau^5 - 4m\tau^3}{(2m - \tau^2)^2} + 2T^2(2T - 4)m\xi^2 \tau^{2T-5} - T^2(2T - 2)\xi^2 \tau^{2T-3}. \quad (7.36)$$

Similarly,

$$\partial_\tau g_{\tau\xi} = 2T(2T - 3)m\xi \tau^{2T-4} - T(2T - 1)\xi \tau^{2T-2}, \quad (7.37)$$

$$\partial_\tau g_{\xi\xi} = 2m(2T - 2)\tau^{2T-3} - 2T\tau^{2T-1}, \quad (7.38)$$

$$\partial_\xi g_{\tau\tau} = 2T^2\xi(2m - \tau^2)\tau^{2T-4}, \quad (7.39)$$

$$\partial_\xi g_{\tau\xi} = T(2m - \tau^2)\tau^{2T-3}, \quad (7.40)$$

and

$$\partial_\xi g_{\xi\xi} = 0. \quad (7.41)$$

To ensure that the expression (7.31) is smooth, we use power counting and try to find a value of T for which it doesn't contain negative powers of τ . The least power of τ in the partial derivatives of the metric is $\min(3, 2T - 5)$, as we can see by inspecting equations (7.36–7.41). The least power of τ in the inverse metric is $\min(-6, -2T + 2)$, as it follows from the equations (7.32–7.34). Since $\min(-6, -2T + 2) = -3 - \max(3, 2T - 5)$, the condition that the least power of τ in (7.31) is non-negative is

$$-1 - 2T + 3 \min(3, 2T - 5) \geq 0 \quad (7.42)$$

with the unique solution

$$T = 4. \quad (7.43)$$

Hence, taking $T = 4$ ensures the smoothness of (7.31), and by this, the semi-regularity of the metric in two dimensions (τ, ξ) .

When going back to four dimensions, we remember the central theorem of semi-regular warped products from [73], stating that the warped product between the two-dimensional extension (τ, ξ) and the sphere S^2 , with warping function τ^2 , is semi-regular. \square

It is useful to extract from the proof the expression of the metric:

Corollary 7.8. *The metric*

$$ds^2 = -\frac{4\tau^4}{2m - \tau^2} d\tau^2 + (2m - \tau^2)\tau^4 (4\xi d\tau + \tau d\xi)^2 + \tau^4 d\sigma^2 \quad (7.44)$$

is an analytic extension of the Schwarzschild metric, which is semi-regular, including at the singularity $r = 0$.

\square

Remark 7.9. The Riemann curvature tensor R_{abcd} is smooth, because the metric is semi-regular. How can it be smooth, when we know that the Riemann curvature of the Schwarzschild metric tends to infinity when it approaches the singularity $r = 0$? The answer is that the coefficients of R_{abcd} depend on the coordinate system. Since the usual coordinates used with the Schwarzschild black hole solution are singular with respect to ours, a tensor which is smooth in our coordinates may appear singular in Schwarzschild coordinates. But this should not be a big surprise, because for the Schwarzschild solution the Ricci tensor is 0, hence the scalar curvature is 0 too, and the Einstein's equation is simply $T_{ab} = 0$. On the other hand, the Kretschmann invariant $R_{abcd}R^{abcd}$ still becomes

infinite at $r = 0$, of course, being a scalar, and therefore remaining unchanged at the coordinate transforms. But it does become infinite only because R^{abcd} becomes infinite.

Corollary 7.10. *The Schwarzschild spacetime is quasi-regular (in any atlas compatible with the coordinates (7.22)).*

Proof. We know from [101] that the Schwarzschild spacetime is semi-regular. Since it is also Ricci flat, it follows that S_{abcd} and E_{abcd} are smooth too. \square

7.1.4 Penrose-Carter coordinates for the semi-regular solution

To move to Penrose-Carter coordinates, we apply the same steps as one usually applies for the Schwarzschild black hole ([5], p. 150-156). More precisely, the lightlike coordinates for the Penrose-Carter diagram are

$$\begin{cases} v'' &= \arctan\left((2m)^{-1/2} \exp\left(\frac{v}{4m}\right)\right) \\ w'' &= \arctan\left(-(2m)^{-1/2} \exp\left(-\frac{w}{4m}\right)\right) \end{cases} \quad (7.45)$$

where v, w are the Eddington-Finkelstein lightlike coordinates [65, 66]

$$\begin{cases} v &= t + r + 2m \ln(r - 2m) \\ w &= t - r - 2m \ln(r - 2m). \end{cases} \quad (7.46)$$

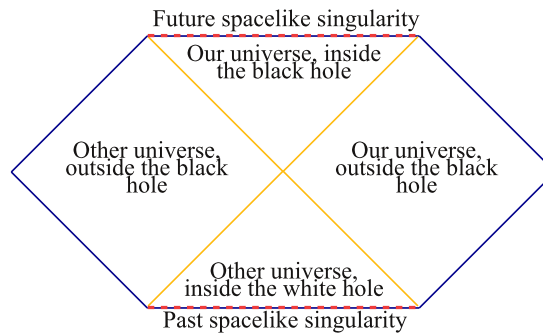


FIGURE 7.1: The maximally extended Schwarzschild solution, in Penrose-Carter coordinates.

Usually, in the Penrose-Carter diagram of the Schwarzschild spacetime is considered that the maximal analytic extension is given by the conditions $v'' + w'' \in (-\pi, \pi)$ and $v'', w'' \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (see Fig. 7.1). This is because we have to stop at the singularity $r = 0$, because the infinite values we get prevent the analytic continuation.

To move from the coordinates (τ, ξ) to the Penrose-Carter coordinates, we use the substitution (7.22):

$$\begin{cases} v &= \xi\tau^4 + \tau^2 + 2m \ln(2m - \tau^2) \\ w &= \xi\tau^4 - \tau^2 - 2m \ln(2m - \tau^2). \end{cases} \quad (7.47)$$

Our coordinates allow us to go beyond the singularity. As we can see from equation (7.44), our solution extends to negative τ as well. From (7.47) we see that it is symmetric with respect to the hypersurface $\tau = 0$. This leads to the Penrose-Carter diagram from Fig. 7.2.

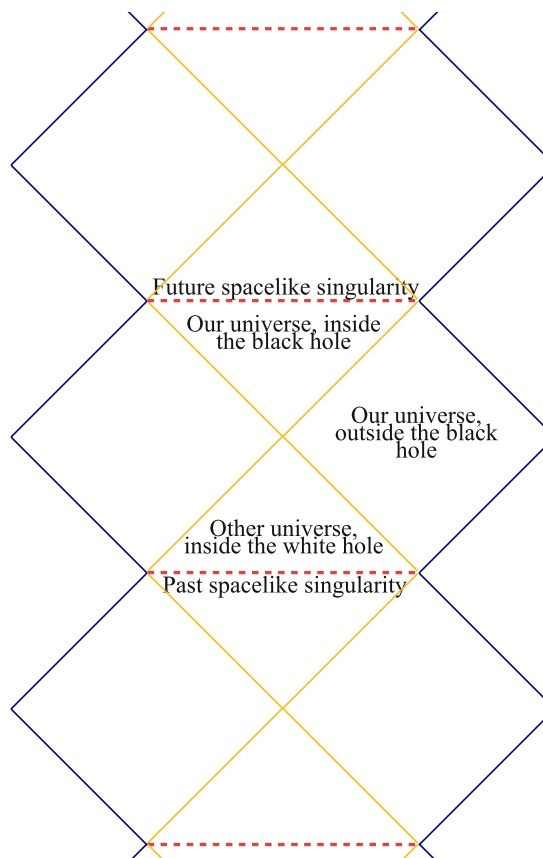


FIGURE 7.2: Our maximally extended Schwarzschild solution, in Penrose-Carter coordinates.

7.1.5 The significance of the semi-regular solution

The main consequence of the extensibility of the Schwarzschild solution to a semi-regular solution beyond the singularity is that the information is not lost there. This can apply as well to the case of an evaporating black hole (see Fig. 7.3).

Because of the no-hair theorem, the Schwarzschild solution is representative for non-rotating and electrically neutral black holes. If the black hole evaporates, the information

reaching the singularity is lost (Fig. 7.3 A). If the singularity is semi-regular, it doesn't destroy the topology of spacetime (Fig. 7.3 B). Moreover, although normally in this case the covariant derivative and other differential operators can't be defined, there is a way to construct them naturally (as shown in [67]), allowing the rewriting of the field equations for the semi-regular case, without running into infinities. This ensures that the field equations can go beyond the singularity.

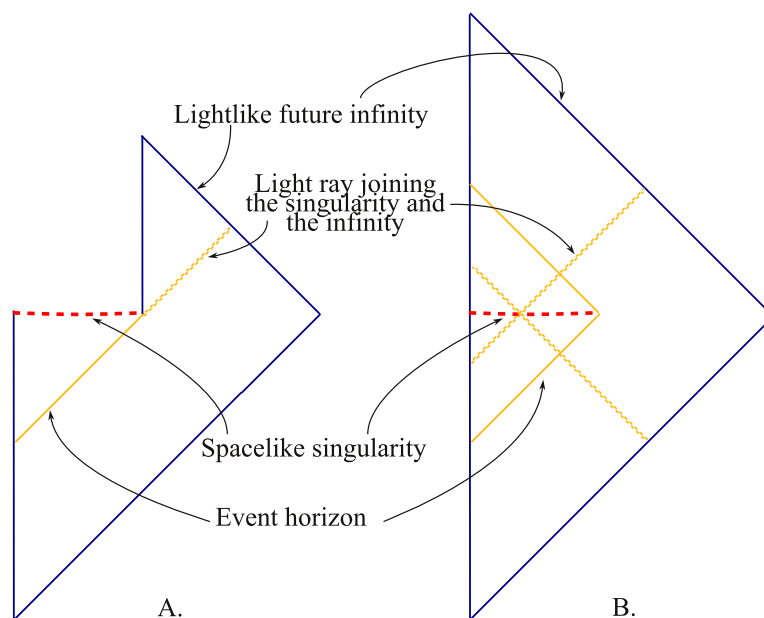


FIGURE 7.3: The Penrose-Carter diagram for a non-rotating and electrically neutral evaporating black hole, whose singularity destroys the information. **A.** Standard evaporating black hole. **B.** Evaporating black hole extended beyond the singularity, whose singularity doesn't destroy the information.

In the case of a black hole which is not primordial and evaporates completely in a finite time, all of the light rays traversing the singularity reach the past and future infinities. This means that the presence of a spacelike evaporating black hole is compatible with the global hyperbolicity, as in the diagram 7.3 B.

The singularity is accompanied by violent and very destructive forces. But, as we can see from the semi-regular formulation, there is no reason to consider that it destroys the information or the structure of spacetime.

7.2 Analytic Reissner-Nordstrom singularity

An analytic extension of the Reissner-Nordström solution at and beyond the singularity is presented. The extension is obtained by using new coordinates in which the metric becomes degenerate at $r = 0$. The metric is still singular in the new coordinates, but its

components become finite and smooth. Using this extension it is shown that the charged and non-rotating black hole singularities are compatible with the global hyperbolicity and with the conservation of the initial value data. Geometric models for electrically charged particles are obtained.

7.2.1 Introduction

7.2.1.1 The Reissner-Nordström solution

The Reissner-Nordström metric describes a static, spherically symmetric, electrically charged, non-rotating black hole [125, 126]. It is a solution to the Einstein-Maxwell equations. It has the following form:

$$ds^2 = - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\sigma^2, \quad (7.48)$$

where q is the electric charge of the body and, as in the case of the Schwarzschild solution,

$$d\sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2 \quad (7.49)$$

is the metric of the unit sphere S^2 , m the mass of the body, and the units were chosen so that $c = 1$ and $G = 1$ (see, *e.g.*, [5], p. 156).

The first two terms in the right hand side of equation (7.48) are independent on the coordinates θ and ϕ , and conversely, $d\sigma^2$ is independent on the coordinates r and t . This solution is a warped product between a two-dimensional ($2D$) semi-Riemannian space and the sphere S^2 with the canonical metric (7.49). Consequently, in coordinate transformations which affect only the coordinates r and t we can ignore the term $r^2 d\sigma^2$ in calculations. We can, finally, reintroduce it, taking again the warped product.

Solving the equation expressing the cancellation of $r^2 - 2mr + q^2$ for r we obtain:

1. no solution, for $q^2 > m^2$ (naked singularity);
2. double solution $r_{\pm} = m$, for $q^2 = m^2$ (the *extremal* case);
3. two solutions $r_{\pm} = m \pm \sqrt{m^2 - q^2}$, for $q^2 < m^2$.

If $q^2 \leq m^2$, there are two singular horizons at $r = r_{\pm}$, which coincide for $q^2 = m^2$. These apparent singularities can be removed by a special coordinate transformation, such as that of Eddington-Finkelstein [65, 66]. All three cases have an irremovable singularity at $r = 0$.

7.2.1.2 Two kinds of metric singularity

There are two main kinds of metric singularities that are relevant for our approach. In the first kind, some of the components of the metric diverge as approaching the singularity, where they become infinite. In the second kind, more benign, the metric remains always smooth, but becomes *degenerate* at the singularity – that is, its determinant becomes 0. In general, this means that the metric is not invertible, *i.e.* g^{ab} tends to infinity and cannot be defined at the singularity. But g_{ab} is smooth, and in some cases, despite the fact that g^{ab} is singular, we can define the contraction between covariant indices, and construct covariant derivatives and the Riemann curvature, and even write an equivalent of the Einstein equation [67, 73, 76].

Some singularities of the first kind, having some components of the metric divergent, can be viewed as singularities of the second kind, expressed in singular coordinate systems. This means that it is possible for some singularities of the first kind to be transformed into singularities of the second kind, by an appropriate choice of the coordinate system. We did this for the Schwarzschild solution in [101].

In this chapter, we will find for the Reissner-Nordström solution a new coordinate system, in which the singularity at $r = 0$ becomes degenerate and analytic. The metric will become degenerate, but all its coefficients will be finite and smooth. The new form of the metric admits an analytic continuation beyond the singularity.

7.2.2 Extending the Reissner-Nordström spacetime at the singularity

7.2.2.1 The main result

The main result of this chapter is contained in the following theorem.

Theorem 7.11. *The Reissner-Nordström metric admits an analytic extension at $r = 0$.*

Proof. We work initially in two dimensions (t, r) . It is enough to make the coordinate transformation in a neighborhood of the singularity – in the region $r \in [0, M)$, where $M = r_-$ if $q^2 \leq m^2$, and $M = \infty$ otherwise. We choose the coordinates ρ and τ , so that

$$\begin{cases} t &= \tau \rho^T \\ r &= \rho^S \end{cases} \quad (7.50)$$

where S, T have to be determined in order to make the metric analytic (figure 7.4). This choice is motivated by the need to stretch the spacetime while approaching the singularity $r = 0$, so that the divergent components of the metric are smoothened.

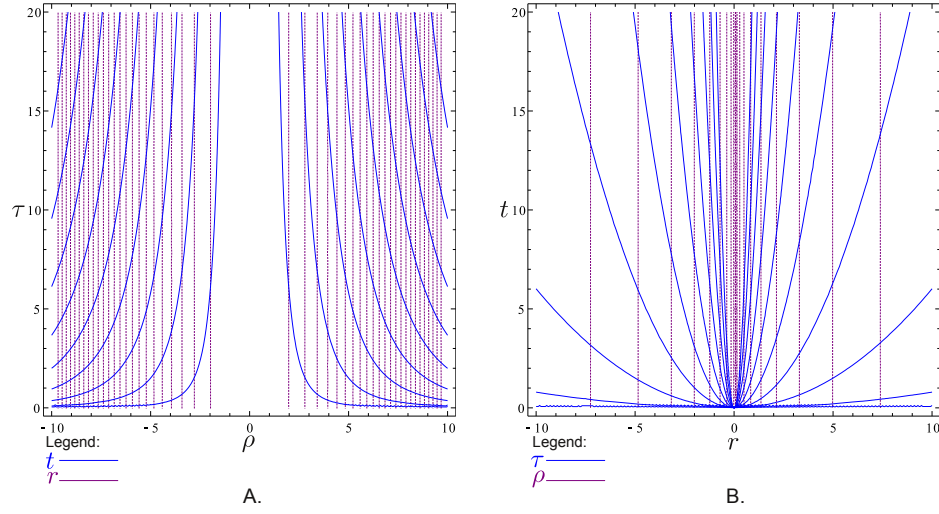


FIGURE 7.4: The coordinate transformations (7.50), represented for $S = 2$ and $T = 4$. A. The coordinates (t, r) expressed in coordinates (τ, ρ) . B. The coordinates (τ, ρ) expressed in coordinates (t, r) .

Then, we have

$$\frac{\partial t}{\partial \tau} = \rho^T, \quad \frac{\partial t}{\partial \rho} = T\tau\rho^{T-1}, \quad \frac{\partial r}{\partial \tau} = 0, \quad \frac{\partial r}{\partial \rho} = S\rho^{S-1}. \quad (7.51)$$

Let us introduce the standard notation

$$\Delta := r^2 - 2mr + q^2 \quad (\text{hence } \Delta = \rho^{2S} - 2m\rho^S + q^2). \quad (7.52)$$

We note that $\Delta > 0$ for $\rho \in [0, M)$.

The metric components in (7.48) become now

$$g_{tt} = -\frac{\Delta}{\rho^{2S}}, \quad g_{rr} = \frac{\rho^{2S}}{\Delta}, \quad g_{tr} = g_{rt} = 0. \quad (7.53)$$

Let us calculate the metric components in the new coordinates.

$$\begin{aligned} g_{\tau\tau} &= \left(\frac{\partial r}{\partial \tau}\right)^2 \frac{\rho^{2S}}{\Delta} - \left(\frac{\partial t}{\partial \tau}\right)^2 \frac{\Delta}{\rho^{2S}} \\ &= 0 - \rho^{2T} \frac{\Delta}{\rho^{2S}} \end{aligned}$$

Therefore

$$\begin{aligned} g_{\tau\tau} &= -\Delta\rho^{2T-2S}. \\ g_{\rho\tau} &= \frac{\partial r}{\partial \rho} \frac{\partial r}{\partial \tau} \frac{\rho^{2S}}{\Delta} - \frac{\partial t}{\partial \rho} \frac{\partial t}{\partial \tau} \frac{\Delta}{\rho^{2S}} \\ &= 0 - T\tau\rho^{2T-1} \frac{\Delta}{\rho^{2S}} \end{aligned} \quad (7.54)$$

Then

$$g_{\rho\tau} = -T\Delta\tau\rho^{2T-2S-1}. \quad (7.55)$$

$$g_{\rho\rho} = S^2\rho^{2S-2}\frac{\rho^{2S}}{\Delta} - T^2\tau^2\rho^{2T-2}\frac{\Delta}{\rho^{2S}}$$

Hence

$$g_{\rho\rho} = S^2\frac{\rho^{4S-2}}{\Delta} - T^2\Delta\tau^2\rho^{2T-2S-2}. \quad (7.56)$$

We can see from the term $S^2\frac{\rho^{4S-2}}{\Delta}$ of $g_{\rho\rho}$ from equation (7.56) that, to ensure that the singularity at $\rho = 0$ is only of degenerate type and the metric is continuous there, S has to be an integer so that $S \geq 1$. Moreover, the condition $S \geq 1$ makes this term analytic at $\rho = 0$, because the denominator does not cancel there and is analytic, and the numerator is analytic.

The other terms of the equation (7.56), and the other equations (7.54) and (7.55), contain as factor Δ , in which the minimum power to which ρ appears is 0. Hence, in order to avoid negative powers of ρ , these terms require that $2T - 2S - 2 \geq 0$. Therefore, the conditions for removing the infinity of the metric at $r = 0$ by a coordinate transformation are that S and T be integers so that:

$$\begin{cases} S \geq 1 \\ T \geq S + 1 \end{cases} \quad (7.57)$$

and they also ensure that the metric is analytic at $r = 0$. None of the metric's components become infinite at the singularity.

To go back to four dimensions, we have to take the warped product between the $2D$ space with the metric we obtained, and the sphere S^2 , with warping function ρ^S . This is a degenerate warped product, as was studied in [73], and its result is a $4D$ manifold whose metric is analytic and degenerate at $\rho = 0$. Hence, this extension of the Reissner-Nordström solution is analytic at $\rho = 0$. \square

Let us extract from the proof the expression of the metric:

Corollary 7.12. *The Reissner-Nordström metric, expressed in the coordinates from theorem 7.11, has the following form*

$$ds^2 = -\Delta\rho^{2T-2S-2}(\rho d\tau + T\tau d\rho)^2 + \frac{S^2}{\Delta}\rho^{4S-2}d\rho^2 + \rho^{2S}d\sigma^2. \quad (7.58)$$

Proof. From (7.51) we find

$$dt = \frac{\partial t}{\partial \tau}d\tau + \frac{\partial t}{\partial \rho}d\rho = \rho^T d\tau + T\tau\rho^{T-1}d\rho = \rho^{T-1}(\rho d\tau + T\tau d\rho) \quad (7.59)$$

and

$$dr = \frac{\partial r}{\partial \tau} d\tau + \frac{\partial r}{\partial \rho} d\rho = S\rho^{S-1} d\rho \quad (7.60)$$

which when plugged in the Reissner-Nordström equation (7.48) give

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^{2S}} dt^2 + \frac{\rho^{2S}}{\Delta} dr^2 + r^2 d\sigma^2 \\ &= -\Delta \rho^{2T-2S-2} (\rho d\tau + T\tau d\rho)^2 + \frac{S^2}{\Delta} \rho^{4S-2} d\rho^2 + \rho^{2S} d\sigma^2. \end{aligned}$$

□

7.2.2.2 The electromagnetic field

The potential of the electromagnetic field in the Reissner-Nordström solution is

$$A = -\frac{q}{r} dt, \quad (7.61)$$

and is singular at $r = 0$ in the standard coordinates (t, r, ϕ, θ) . On the other hand, in the new coordinates it is smooth.

Corollary 7.13. *In the new coordinates $(\tau, \rho, \phi, \theta)$, the electromagnetic potential is*

$$A = -q\rho^{T-S-1} (\rho d\tau + T\tau d\rho), \quad (7.62)$$

the electromagnetic field is

$$F = q(2T - S)\rho^{T-S-1} d\tau \wedge d\rho, \quad (7.63)$$

and they are analytic everywhere, including at the singularity $\rho = 0$.

Proof. The equation of the electromagnetic potential follows directly from the proof of theorem 7.11 and from equation (7.59). The equation of the electromagnetic field is obtained by applying the exterior derivative:

$$\begin{aligned} F &= dA = -q d(\rho^{T-S} d\tau + T\tau \rho^{T-S-1} d\rho) \\ &= -q \left(\frac{\partial \rho^{T-S}}{\partial \rho} d\rho \wedge d\tau + T \frac{\partial \tau \rho^{T-S-1}}{\partial \tau} d\tau \wedge d\rho \right) \\ &= -q ((T - S)\rho^{T-S-1} d\rho \wedge d\tau + T\rho^{T-S-1} d\tau \wedge d\rho) \\ &= q(2T - S)\rho^{T-S-1} d\tau \wedge d\rho \end{aligned}$$

□

7.2.2.3 General remarks concerning the proposed degenerate extension

Remark 7.14. The analytic Reissner-Nordström solution we found extends through $\rho = 0$ to negative values of ρ . If S is even, ρ and $-\rho$ give the same metric. For even values of T , also the electromagnetic potential is invariant at the space inversion $\rho \mapsto -\rho$. After taking the warped product we can identify the points $(\tau, \rho, \phi, \theta)$ and $(\tau, -\rho, (\phi + \pi) \bmod 2\pi, \theta)$, and also all the points from the warped product which have $\rho = 0$ and constant τ . This identification gives a smooth metric, because of the symmetry with respect to the axis $\rho = 0$, and because the warping function is ρ^S , with $S \geq 1$. We obtain by this a spherically symmetric solution having the topology of \mathbb{R}^4 .

If we choose not to make this identification, the extension through $\rho = 0$ looks like the Einstein-Rosen model of charged particles [40], or like Misner and Wheeler’s “charge without charge” [127]. As is known from the “charge without charge” program, special topology (*i.e.* “wormholes”) allows the existence of source-free electromagnetic fields which look as being associated to charges, without actually having sources. The proposed degenerate extension of the Reissner-Nordström spacetime seems to support these proposals, but by making the above-mentioned identification, it also allows charge models with the standard \mathbb{R}^4 topology.

If S is odd, the extension to $\rho < 0$ is very similar to the extension from the Kerr and Kerr-Newmann solutions through the interior of the ring singularity to the region $r < 0$.

Remark 7.15. As in the case of the analytic extension of the Schwarzschild solution [101], there is no unique way to extend the Reissner-Nordström metric so that it is smooth at the singularity. The explanation is due to the fact that a degenerate metric can remain smooth and even analytic at certain singular coordinate transformations.

A *semi-regular* metric has smooth Riemann curvature R_{abcd} , and allows the construction of more useful operations which are normally prohibited by the fact that the metric is degenerate. In the case of the Schwarzschild black hole we could find a solution which is semi-regular [101]. In the case of the Reissner-Nordström black hole, we can’t find numbers S and T for the equation (7.50), which would make the metric semi-regular. However, this does not exclude other changes of the coordinates, and we propose the following open problem:

Open Problem 7.16. Is it possible to find coordinates which allow the Reissner-Nordström metric to be semi-regular at $\rho = 0$?

Also, it may be interesting the following:

Open Problem 7.17. Can we find natural conditions ensuring the uniqueness of the analytic extensions of the Schwarzschild and Reissner-Nordström solutions at the singularity $\rho = 0$? Under what conditions does a singular coordinate transformation of an analytic extension lead to another extension which is physically indistinguishable?

7.2.3 Null geodesics in the proposed solution

In this section, we will discuss the geometric meaning of the extension proposed in this chapter, mainly from the viewpoint of the lightcones and the null geodesics. In the coordinates (τ, ρ) , the metric is analytic near the singularity $\rho = 0$ and has the form

$$g = -\Delta \rho^{2T-2S-2} \begin{pmatrix} \rho^2 & T\tau\rho \\ T\tau\rho & T^2\tau^2 - \frac{S^2}{\Delta^2}\rho^{6S-2T} \end{pmatrix} \quad (7.64)$$

Let us find the null directions, defined at each point (τ, ρ) by the tangent vectors $u \neq 0$ so that $g(u, u) = 0$. Since any nonzero multiple of u is also a solution, we will consider $u = (\sin \alpha, \cos \alpha)$, and try to find α . We obtain the equation

$$\rho^2 \sin^2 \alpha + 2T\tau\rho \sin \alpha \cos \alpha + \left(T^2\tau^2 - \frac{S^2}{\Delta^2}\rho^{6S-2T} \right) \cos^2 \alpha = 0, \quad (7.65)$$

which can be written as a quadratic equation in $\tan \alpha$

$$\rho^2 \tan^2 \alpha + 2T\tau\rho \tan \alpha + \left(T^2\tau^2 - \frac{S^2}{\Delta^2}\rho^{6S-2T} \right) = 0, \quad (7.66)$$

which leads to the solution

$$\tan \alpha_{\pm} = -\frac{T\tau}{\rho} \pm \frac{S}{\Delta} \rho^{3S-T-1}. \quad (7.67)$$

Therefore, the incoming and outgoing null geodesics satisfy the differential equation

$$\frac{d\tau}{d\rho} = -\frac{T\tau}{\rho} \pm \frac{S}{\Delta} \rho^{3S-T-1}. \quad (7.68)$$

The coordinate ρ remains spacelike only as long as $g_{\rho\rho} > 0$, and from equation (7.64) we can see that this requires that

$$\frac{S^2}{\Delta^2} \rho^{6S-2T} > T^2 \tau^2. \quad (7.69)$$

To ensure the condition (7.69) in a neighborhood of $(0,0)$, we need to choose T so that

$$T \geq 3S. \quad (7.70)$$

The null geodesics are the integral curves of the null vectors found in (7.67). We see that, in the coordinates (τ, ρ) , the null geodesics are oblique everywhere, except at $\rho = 0$, where they become tangent to the axis defined by $\rho = 0$. Hence, the degeneracy of the metric is expressed by the fact that the lightcones stretch as approaching $\rho = 0$, where they become degenerate (figure 7.5). At these points, the incoming null geodesics become tangent to the outgoing null geodesics (figure 7.6).

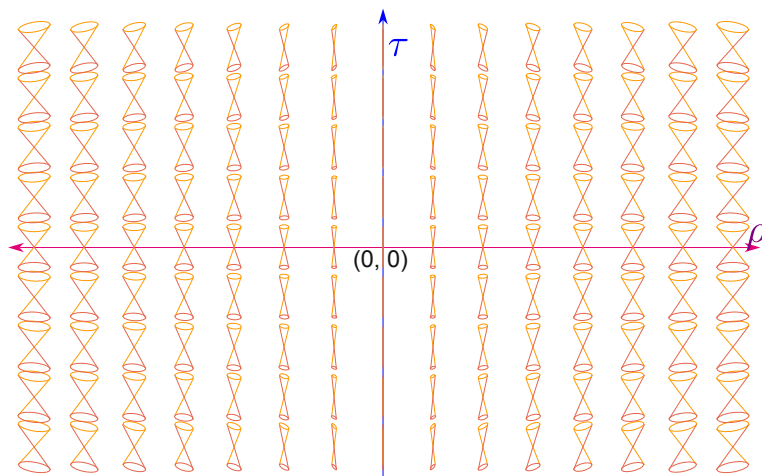


FIGURE 7.5: As one approaches the singularity on the axis $\rho = 0$, the lightcones become more and more degenerate along that axis (for $T \geq 3S$ and even S).

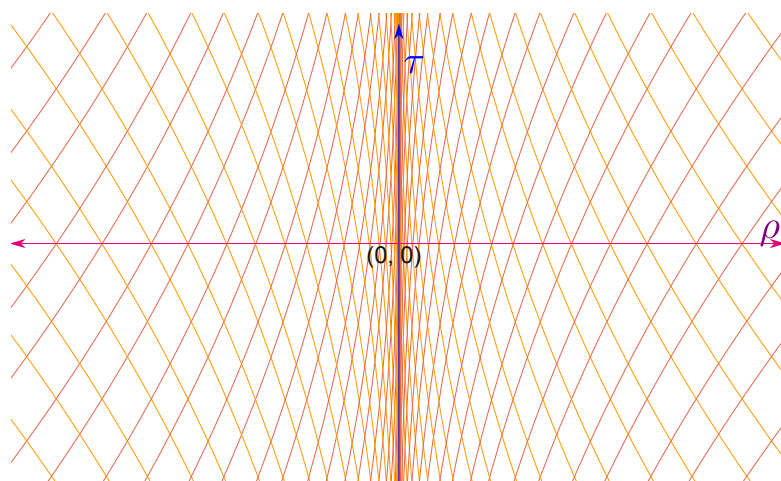


FIGURE 7.6: The null geodesics, in the (τ, ρ) coordinates, for $T \geq 3S$ and even S .

7.2.4 The Penrose-Carter diagrams for our solution

To move to Penrose-Carter coordinates (and have a bird's eye view of the global behavior of the degenerate extensions of the Reissner-Nordström solution), we apply the same steps as those one normally applies for the standard Reissner-Nordström black hole. These steps are, for example, presented in [5], p. 157-161, and lead from the coordinates (t, r) to the Penrose-Carter coordinates (figure 7.7).

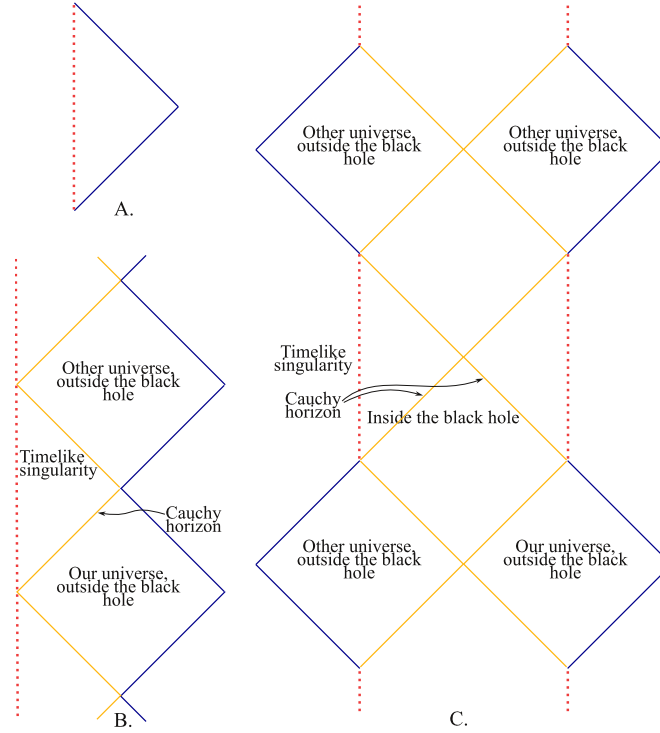


FIGURE 7.7: A. Naked Reissner-Nordström black holes ($q^2 > m^2$). B. Extremal Reissner-Nordström black holes ($q^2 = m^2$). C. Reissner-Nordström black holes with $q^2 < m^2$.

We just add our coordinate transformation before the steps leading to the Penrose-Carter coordinates, as we did in [101] for the Schwarzschild solution. If S is odd, the spacetime has a region $\rho < 0$ and the Penrose-Carter diagrams are similar to the standard diagrams for the Kerr and Kerr-Newman spacetimes (see for example [5], p. 165). If S is even, the diagram will repeat not only vertically, but also horizontally, symmetrical to the singularity.

We obtain the diagram for the naked Reissner-Nordström black hole ($q^2 > m^2$) by taking the symmetric of the standard Reissner-Nordström diagram with respect to the singularity (figure 7.8).

The resulting diagram for the extremal Reissner-Nordström black hole ($q^2 = m^2$) is a strip symmetric about the singularity (figure 7.9).

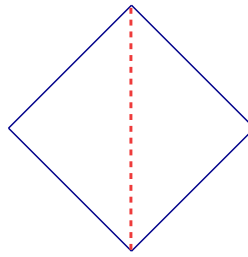


FIGURE 7.8: Penrose-Carter diagram for the naked Reissner-Nordström black hole ($q^2 > m^2$), analytically extended beyond the singularity. It is symmetric with respect to the timelike singularity.

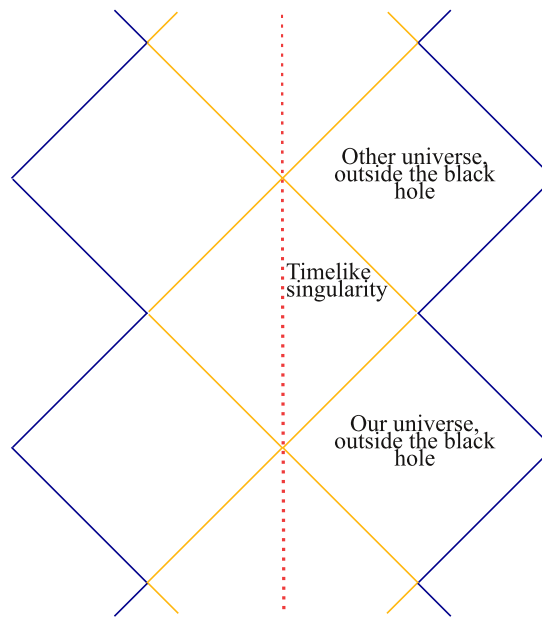


FIGURE 7.9: Penrose-Carter diagram for the extremal Reissner-Nordström black hole ($q^2 = m^2$), analytically extended beyond the singularity. It repeats periodically along the vertical direction.

When represented in plane, the diagram for the non-extremal Reissner-Nordström black hole ($q^2 < m^2$) extends in two directions and has overlapping parts (figure 7.10).

In the Penrose-Carter diagrams of the degenerate extension of the Reissner-Nordström solution the null geodesics continue through the singularity, because they are always at $\pm \frac{\pi}{4}$.

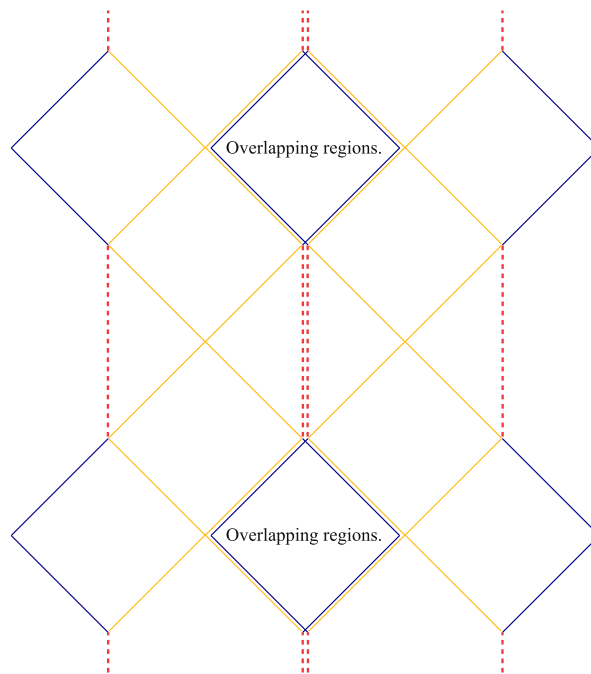


FIGURE 7.10: The Penrose-Carter diagram for the non-extremal Reissner-Nordström black hole with $q^2 < m^2$, analytically extended beyond the singularity. When represented in plane, it repeats periodically along both the vertical and the horizontal directions, and it has overlaps. In the diagram, there is a small shift between the two copies, to make the overlapping visible.

7.2.5 A globally hyperbolic charged black hole

A global solution to the Einstein equation is well-behaved when the equations at a given moment of time determine the solution for the entire future and past. This condition is ensured by the *global hyperbolicity*, which is expressed by the requirements that

1. for any two points p and q , the intersection between the causal future of p , and the causal past of q , $J^+(p) \cap J^-(q)$, is a compact subset of the spacetime;
2. there are no closed timelike curves ([5], p. 206).

The property of global hyperbolicity is equivalent to the existence of a *Cauchy hypersurface* – a spacelike hypersurface \mathfrak{S} that, for any point p in the future (past) of \mathfrak{S} , is intersected by all past-directed (future-directed) inextendible causal (*i.e.* timelike or null) curves through the point p ([5], p. 119, 209–212).

Because in the standard coordinates for the Reissner-Nordström spacetime one cannot extend the solution beyond the singularity, the Reissner-Nordström spacetime fails to

admit a Cauchy hypersurface, and it is normally inferred that it is not globally hyperbolic.

But since we now know how to extend analytically the Reissner-Nordström spacetime beyond the singularity, we should check if we can use this feature to construct new solutions which are globally hyperbolic. To do so, we will construct solutions that admit *foliations* with Cauchy hypersurfaces – *i.e.* that are diffeomorphic with a Cartesian product between an interval $I \subseteq \mathbb{R}$ representing the time dimension, and a spacelike hypersurface.

The coordinates (τ, ρ) , under the condition (7.70), provide a spacelike foliation given by the hypersurfaces $\tau = \text{const.}$ This foliation is global only for naked singularities; otherwise it is defined locally, in a neighborhood of $(\tau, \rho) = (0, 0)$ given by $r < r_-$. From the equation (7.48) defining the Reissner-Nordström metric we know that the solution is *stationary*; that is, when expressed in the coordinates (t, r) it is independent of time. This means that we can choose as the origin of time any value, this ensuring that we can cover a neighborhood of the entire axis $\rho = 0$ with coordinate patches like (τ, ρ) . To obtain global foliations with Cauchy hypersurfaces, we use the global extensions represented in the Penrose-Carter diagrams of section §7.2.4, figures 7.8, 7.9 and 7.10. It is important to note that in the Penrose-Carter diagram, the null directions are represented as straight lines inclined at $\pm \frac{\pi}{4}$.

For the naked Reissner-Nordström solution (figure 7.8) we can find immediately a global foliation, because the Penrose-Carter diagram is identical to that for the Minkowski spacetime (figure 7.11). Hence, the natural foliation of the Minkowski spacetime will be good for our extended naked Reissner-Nordström solution too.

To obtain explicitly the foliations for all the cases, we map to our solutions represented in coordinates (τ, ρ) the product $(0, 1) \times \mathbb{R}$. To do this, we can use a version of the Schwarz-Christoffel mapping that maps the strip

$$\mathcal{S} := \{z \in \mathbb{C} | \text{Im}(z) \in [0, 1]\} \quad (7.71)$$

to a polygonal region from \mathbb{C} , with the help of the formula

$$f(z) = A + C \int^{\mathcal{S}} \exp \left[\frac{\pi}{2} (\alpha_- - \alpha_+) \zeta \right] \prod_{k=1}^n \left[\sinh \frac{\pi}{2} (\zeta - z_k) \right]^{\alpha_k - 1} d\zeta, \quad (7.72)$$

where $z_k \in \partial \mathcal{S} := \mathbb{R} \times \{0, i\}$ are the prevertices of the polygon, and $\alpha_-, \alpha_+, \alpha_k$ are the measures of the angles of the polygon, divided by π (*cf. e.g.* [128]). The vertices having the angles α_- and α_+ correspond to the ends of the strip, which are at infinity. The level curves $\{\text{Im}(z) = \text{const.}\}$ give our foliation [102].

The prevertices whose image is represented in Figure 7.14 are

$$(-\infty, 0, +\infty, i), \quad (7.73)$$

and the angles are

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right). \quad (7.74)$$

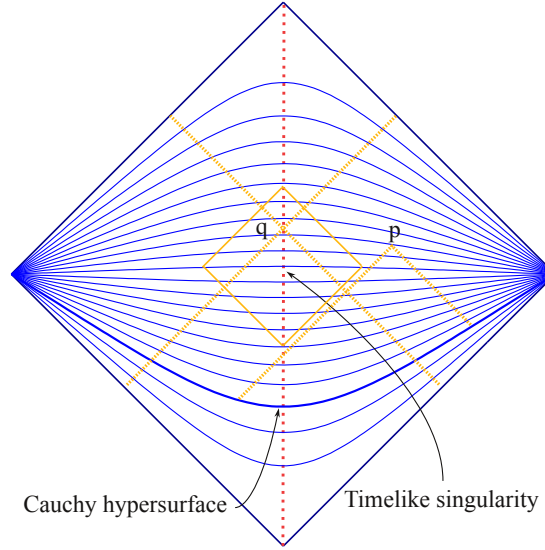


FIGURE 7.11: Spacelike foliation of the naked Reissner-Nordström solution ($q^2 > m^2$). The spacelike hypersurfaces are Cauchy. Every point of the singularity can be joined with the future and past null infinities.

For the other cases $q^2 \leq m^2$ (figure 7.7 (B) and (C)) the maximal extensions cannot be globally hyperbolic, because they admit *Cauchy horizons* (hypersurfaces which are boundaries for the Cauchy development of the data on a spacelike hypersurface). If we want to obtain a globally hyperbolic solution, we have to drop the regions beyond the Cauchy horizons. This leads naturally to a choice of a subset of the Penrose-Carter diagram which is symmetric about the singularity $r = 0$ and can be foliated (Figures 7.12 and 7.13).

Let us take now as prevertices of the Schwarz-Christoffel mapping (7.72) the set

$$(-\infty, -a, 0, a, +\infty, i), \quad (7.75)$$

where $0 < a$ is a positive real number. The angles are, respectively

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right). \quad (7.76)$$

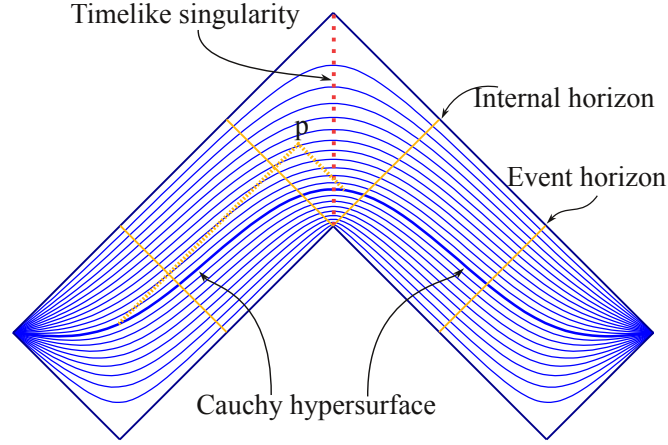


FIGURE 7.12: Foliation of the non-extremal Reissner-Nordström solution ($q^2 < m^2$), with Cauchy hypersurfaces.

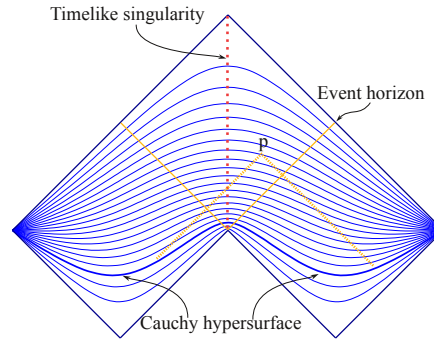


FIGURE 7.13: Foliation of the extremal Reissner-Nordström solution with $q^2 = m^2$, with Cauchy hypersurfaces.

Appropriate choices of a result in the foliations represented in diagrams 7.12 and 7.13, corresponding to the non-extremal, respectively the extremal solutions with $q^2 < m^2$. Since $\alpha_- = \alpha_+$ and the edges are inclined at most at $\frac{\pi}{4}$, alternating in such a way that the level curves with $\text{Im}(z) \in (0, 1)$ have at each point tangents making an angle strictly between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, our foliations are spacelike.

In each of figures 7.11, 7.12, and 7.13 we highlighted a spacelike hypersurface which is Cauchy, because it is intersected by all past (future) directed inextensible causal curves through a point p from its future (past).

7.2.6 The meaning of the analytic extension at the singularity

As in the case of the extension of the Schwarzschild solution, we can see that the singularity is not necessarily harmful for the information or the structure of spacetime. There is no reason to believe that the information is lost at the singularity, and the fact that

it is timelike and may be naked, although it contradicts Penrose's cosmic censorship hypothesis, is compatible with the global hyperbolicity. These observations may apply also to the case of an evaporating charged black hole (see figure 7.14).

The Reissner-Nordström solution is, according to the no-hair theorem, representative of non-rotating and electrically charged black holes. If the black hole evaporates, the singularity becomes visible to the distant observers. This is a problem in the solutions which do not admit extension through the singularity. Our solution, because it can be extended beyond the singularity, does not break the topology of spacetime. The metric tensor does not run into infinities, although, because of its degeneracy, other quantities, such as its inverse, may become infinite.

The maximal globally hyperbolic extensions from section §7.2.5 are ideal, because the Reissner-Nordström solutions describe spacetimes which are too simple. But because they can be foliated by Cauchy hypersurfaces, and the base hypersurface is \mathbb{R}^3 , we can interpolate between such solutions and foliations without singularities, and construct more general solutions. The interpolation can be done by varying the parameters m and q . By this, one can model spacetimes with black holes that are formed and then evaporate. The presence of a timelike evaporating singularity of this type is compatible with the global hyperbolicity, as in figure 7.14.

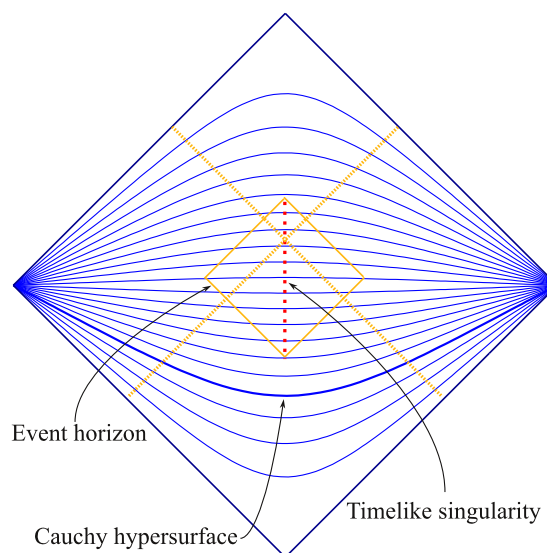


FIGURE 7.14: Non-primordial evaporating black hole with timelike singularity. The fact that the points of the singularity become visible to distant observers is not a problem for the global hyperbolicity, because the null geodesics can be extended beyond the singularity.

This extension of the Reissner-Nordström solution can be used to model electrically charged particles as charged black holes, as pointed out in Remark 7.14.

7.3 Kerr-Newman solutions with analytic singularity

It is shown that the Kerr-Newman solution, representing charged and rotating stationary black holes, admits analytic extension at the singularity. This extension is obtained by using new coordinates, in which the metric tensor becomes smooth on the singularity ring. On the singularity, the metric is degenerate - its determinant cancels. The analytic extension can be naturally chosen so that the region with negative r no longer exists, eliminating by this the closed timelike curves normally present in the Kerr and Kerr-Newman solutions. On the extension proposed here the electromagnetic potential is smooth, being thus able to provide non-singular models of charged spinning particles. The maximal analytic extension of this solution can be restrained to a globally hyperbolic region containing the exterior universe, having the same topology as the Minkowski spacetime. This admits a spacelike foliation in Cauchy hypersurfaces, on which the information contained in the initial data is preserved.

7.3.1 Introduction

The Kerr-Newman solutions are stationary and axisymmetric solutions of the Einstein-Maxwell equations, representing charged rotating black holes [117, 129].

The other stationary black hole solutions can be obtained as particular cases of the Kerr-Newman solutions. They are representative for all the black holes, because even the non-stationary black holes tend in time to Kerr-Newman ones (according to the no-hair theorem).

But they also have some unusual properties, which are in general considered undesirable. They, as any black hole solution, have a singularity, where some of the fields reach infinite values. The singularity is in general ring-shaped, and passing through the ring one can reach inside another universe, in which there are *closed timelike curves*, *i.e.* time machines (which fortunately don't affect the causality in the region $r > 0$)¹. But there is also another problem, the *black hole information paradox*, which refers to the loss of information inside the singularity, which, if would really happen, would cause serious problems, especially violation of unitary evolution, after the black hole evaporation [78, 82].

The metric can be singular in two main ways which are relevant to our discussion. In the first kind of singularity, there are components of the metric which diverge as approaching the singularity. The Kerr-Newman metric is, in usual coordinates, of the first kind. The

¹The existence of closed timelike curves in the region $r < 0$ of the Kerr-Newman spacetime seems to depend on the coordinate system [130].

second kind is that when the metric's components remain smooth at the singularity (and therefore finite). In the second kind, the singularity is still present², because the metric becomes *degenerate* – *i.e.* its determinant becomes 0. In some cases, it is possible to change the coordinate system in which a singularity of the first kind is represented, so that in the new coordinates the singularity becomes of the second kind – it becomes degenerate.

The purpose of this chapter is to show that there are coordinates in which the singularity of the Kerr-Newman metric becomes of degenerate type. In these coordinates, the metric becomes smooth, and the only way the singularity manifests is that the metric becomes degenerate (we have already developed, in [67, 73, 76], mathematical tools which allow us to make differential geometry even in this situation of degenerate metric). In addition, we will show here that we can choose the analytic extension so that the closed timelike curves no longer exist. Moreover, we can find solutions which are globally hyperbolic and admit spacelike foliations in Cauchy hypersurfaces, ensuring therefore the conservation of information. The electromagnetic potential turns out to be smooth. New models for charged spinning particles are suggested.

The Kerr-Newman metric is usually defined in $\mathbb{R} \times \mathbb{R}^3$, where \mathbb{R} is the time coordinate, and on \mathbb{R}^3 we use spherical coordinates (r, ϕ, θ) . Let $a \geq 0$ (which characterizes the rotation), $m \geq 0$ the mass, $q \in \mathbb{R}$ the charge, and let's define the functions

$$\Sigma(r, \theta) := r^2 + a^2 \cos^2 \theta \quad (7.77)$$

and

$$\Delta(r) := r^2 - 2mr + a^2 + q^2. \quad (7.78)$$

Then, we define the Kerr-Newman metric by

$$g_{tt} = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \quad (7.79)$$

$$g_{rr} = \frac{\Sigma}{\Delta} \quad (7.80)$$

$$g_{\phi\phi} = \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta \quad (7.81)$$

$$g_{t\phi} = g_{\phi t} = -\frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} \quad (7.82)$$

$$g_{\theta\theta} = \Sigma \quad (7.83)$$

²It may happen that the metric becomes regular after the coordinate transformation, but in this case it follows that the singularity was not genuine, it was due to the fact that the coordinates in which the regular metric was represented are singular. This is the case of the Eddington-Finkelstein coordinates [65, 66], which proved that the singularity of the event horizon is only apparent.

all other components of the metric being equal to 0 [117].

By making $q = 0$ we obtain the Kerr solution [131, 132], while by making $a = 0$ we get the Reissner-Nordström solution [125, 126]. By making both $q = 0$ and $a = 0$ we obtain the Schwarzschild solution, which when $m = 0$ gives the empty Minkowski spacetime (see Table 7.1).

	$\mathbf{a} > \mathbf{0}$	$\mathbf{a} = \mathbf{0}$
$\mathbf{q} \neq \mathbf{0}$	Kerr-Newman	Reissner-Nordström
$\mathbf{q} = \mathbf{0}$	Kerr	Schwarzschild

TABLE 7.1: The various stationary black hole solutions, as particularizations of the Kerr-Newman solution.

7.3.2 Extending the Kerr-Newman spacetime at the singularity

Theorem 7.18. *The Kerr-Newman metric admits an analytic extension at $r = 0$ (where the metric is degenerate, having analytic and not singular components).*

Proof. We will find a coordinate system in which the metric is analytic, although degenerate. Recall that the event horizons of the black hole are given by the real solutions r_{\pm} of the equation $\Delta = 0$. It is enough to make the coordinate change in a neighborhood of the singularity – in the block III, as it is usually called ([77], p. 66). This is the region $r < r_-$ if r_- is a real (and positive) number. If $\Delta = 0$ has no real solutions, the singularity is naked, and we can take the entire domain.

We choose the coordinates τ , ρ , and μ , so that

$$\begin{cases} t &= \tau \rho^T \\ r &= \rho^S \\ \phi &= \mu \rho^M \end{cases} \quad (7.84)$$

with S, T to be determined in order to make the metric analytic. The expression of the metric tensor when passing from coordinates (x^a) to the new coordinates $(x^{a'})$ is given by:

$$g_{a'b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} g_{ab} \quad (7.85)$$

where Einstein's summation convention is used. In our case, we have the following Jacobian for the coordinate transformation:

$$\frac{\partial(t, r, \phi, \theta)}{\partial(\tau, \rho, \mu, \theta)} = \begin{pmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial \rho} & \frac{\partial t}{\partial \mu} & \frac{\partial t}{\partial \theta} \\ \frac{\partial r}{\partial \tau} & \frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \mu} & \frac{\partial r}{\partial \theta} \\ \frac{\partial \phi}{\partial \tau} & \frac{\partial \phi}{\partial \rho} & \frac{\partial \phi}{\partial \mu} & \frac{\partial \phi}{\partial \theta} \\ \frac{\partial \theta}{\partial \tau} & \frac{\partial \theta}{\partial \rho} & \frac{\partial \theta}{\partial \mu} & \frac{\partial \theta}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \rho^T & T\tau\rho^{T-1} & 0 & 0 \\ 0 & S\rho^{S-1} & 0 & 0 \\ 0 & M\mu\rho^{M-1} & \rho^M & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.86)$$

Let's arrange its coefficients in a table:

	$\cdot/\partial\tau$	$\cdot/\partial\rho$	$\cdot/\partial\mu$	$\cdot/\partial\theta$
$\partial t/\cdot$	ρ^T	$T\tau\rho^{T-1}$	0	0
$\partial r/\cdot$	0	$S\rho^{S-1}$	0	0
$\partial \phi/\cdot$	0	$M\mu\rho^{M-1}$	ρ^M	0
$\partial \theta/\cdot$	0	0	0	1

TABLE 7.2: The Jacobian coefficients of the coordinate change.

We want to make sure that the new expression of the metric becomes smooth even on the ring singularity. For this, we want that all the terms in the right hand side of equation (7.85) are smooth. To ensure this, we have to make sure that the Jacobian coefficients cancels the singularities of the metric components, even when $\cos\theta = 0$.

The least power of ρ on the ring singularity, in each of the metric components listed in equations (7.79), (7.81), and (7.82) are respectively:

$$\mathcal{O}_\rho(g_{tt}) = -2S \quad (7.87)$$

$$\mathcal{O}_\rho(g_{\phi\phi}) = -2S \quad (7.88)$$

$$\mathcal{O}_\rho(g_{t\phi}) = \mathcal{O}_\rho(g_{\phi t}) = -2S \quad (7.89)$$

these components being obtained by dividing polynomial expressions in ρ by Σ . None of the other components can become singular on the ring singularity.

The least power of ρ in each of the Jacobians coefficients from Table 7.2 are given in Table 7.3.

Let's take the metric components and see if they are canceled by the coefficients of the Jacobian.

We check each component g_{ab} of the metric tensor by looking up the rows labeled by $\partial x^a/\cdot$ and $\partial x^b/\cdot$ in Table 7.3.

	$\cdot/\partial\rho$	$\cdot/\partial\tau$	$\cdot/\partial\mu$	$\cdot/\partial\theta$
$\partial t/\cdot$	T	$T - 1$	0	0
$\partial r/\cdot$	0	$S - 1$	0	0
$\partial\phi/\cdot$	0	$M - 1$	M	0
$\partial\theta/\cdot$	0	0	0	0

TABLE 7.3: The least power of ρ in the Jacobian coefficients of the coordinate change.

For example, the term

$$\frac{\partial t}{\partial\rho} \frac{\partial t}{\partial\tau} g_{tt} \quad (7.90)$$

satisfies

$$\mathcal{O}_\rho \left(\frac{\partial t}{\partial\rho} \frac{\partial t}{\partial\tau} g_{tt} \right) = (T - 1) + T - 2S \quad (7.91)$$

hence T needs to satisfy $2T \geq 2S + 1$.

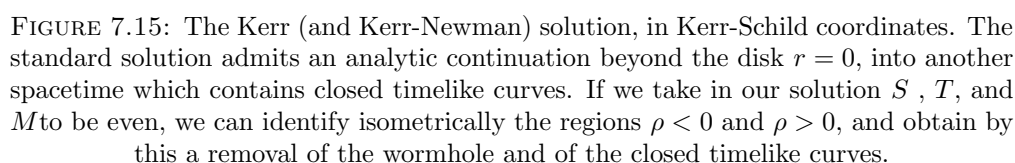
From equations (7.87), (7.88), and (7.89) is easy to see that we have to do this only for the components of the metric with indices t and ϕ . From the equation (7.85) we see that we are interested only in the rows $\partial t/\cdot$ and $\partial\phi/\cdot$ from the Table 7.3. It follows then that each of the coefficients of the Jacobian having the form $\partial t/\cdot$ and $\partial\phi/\cdot$ has to contain ρ to at least the power S , to cancel the metric components. It follows that the conditions

$$\begin{cases} S & \geq 1 \\ T & \geq S + 1 \\ M & \geq S + 1 \end{cases} \quad (7.92)$$

where $S, T, M \in \mathbb{N}$, ensure the smoothness (and the analyticity for that matter) of the metric on the ring singularity, in the new coordinates. None of the metric components in the new coordinates become infinite at the singularity. \square

Remark 7.19. The Kerr-Newman solution has a ring singularity, where $r = 0$ and $\cos\theta = 0$. By using Kerr-Schild coordinates, we can see that it can be analytically extended through the disk defined by $r = 0$ to another spacetime region which looks similar, but is not isometric to the region with $r > 0$, since there $r < 0$ (see Fig. 7.15). On the other hand it is easy to check that, if we use our coordinates with even S , T , and M , then the analytic extension to $\rho < 0$ gives a region which is isometric to that with $\rho > 0$, with the isometry given by identifying the points $(\rho, \tau, \mu, \theta)$ and $(-\rho, \tau, \mu, \theta)$.

Remark 7.20. Our global solution described in the Remark 7.19 shows that, for even S , T , and M , we can eliminate the region where $r < 0$. In this case, the closed timelike curves known to appear in the standard Kerr and Kerr-Newman solutions, are no longer present. Therefore, if these closed timelike curves were considered as violating the causality, to avoid them we just take S , T , and M to be even and make the identification of $(\rho, \tau, \mu, \theta)$ and $(-\rho, \tau, \mu, \theta)$.



7.3.3 The electromagnetic field

One distinctive feature of our extension is that it has smooth electromagnetic potential and electromagnetic field. This may be important in particular when using the Kerr-Newman black holes to model charged particles.

The electromagnetic potential of the Kerr-Newman solution is

$$A = -\frac{qr}{\Sigma}(dt - a \sin^2 \theta d\phi) \quad (7.93)$$

which becomes in our coordinates

$$A = -\frac{q\rho^S}{\Sigma}(\rho^T d\tau + T\tau\rho^{T-1}d\rho - a \sin^2 \theta \rho^M d\mu) \quad (7.94)$$

because from the Table 7.2 it follows that

$$dt = \rho^T d\tau + T\tau\rho^{T-1}d\rho \quad (7.95)$$

$$dr = S\rho^{S-1}d\rho \quad (7.96)$$

and

$$d\phi = M\mu\rho^{M-1}d\rho + \rho^M d\mu \quad (7.97)$$

The singularity of the electromagnetic potential A at $\rho = 0$ and $\cos \theta = 0$ is removed in our case, since $T > S$ and $M > S$, from the conditions (7.92). Similarly, since $F = dA$, we conclude that the electromagnetic field is smooth too.

7.3.4 The global solution

The Penrose-Carter diagrams of our solution depend on the various combinations of the parameters a, q, m . For the Schwarzschild solution they were presented in [101], and for the Reissner-Nordström in [96]. In general it is admitted that the Kerr and Kerr-Newman solutions have Penrose-Carter diagrams similar to those for the Reissner-Nordström solution, although there are some differences due to the fact that the symmetry is not spherical, but axisymmetric, that the singularity is ring-shaped, and of the closed time-like curves in the region $r < 0$. Since our solution can eliminate the closed timelike curves (Remark 7.19), we expect a better similarity with the Reissner-Nordström case, and consequently similar Penrose-Carter diagrams. This would allow similar spacelike foliations of the spacetime as those presented in [96] for the Reissner-Nordström case, except that the singularity is ring-shaped (see Figure 7.16). The foliations are obtained exactly as in the Reissner-Nordström case [96], by using the same Schwarz-Christoffel mappings. As in that case, to obtain maximal globally hyperbolic extensions, we don't take the maximal analytic continuations of the solutions for $a^2 + q^2 \geq m^2$ beyond the Cauchy horizons. To avoid these horizons, we limit the foliations to globally hyperbolic regions containing the exterior universe.

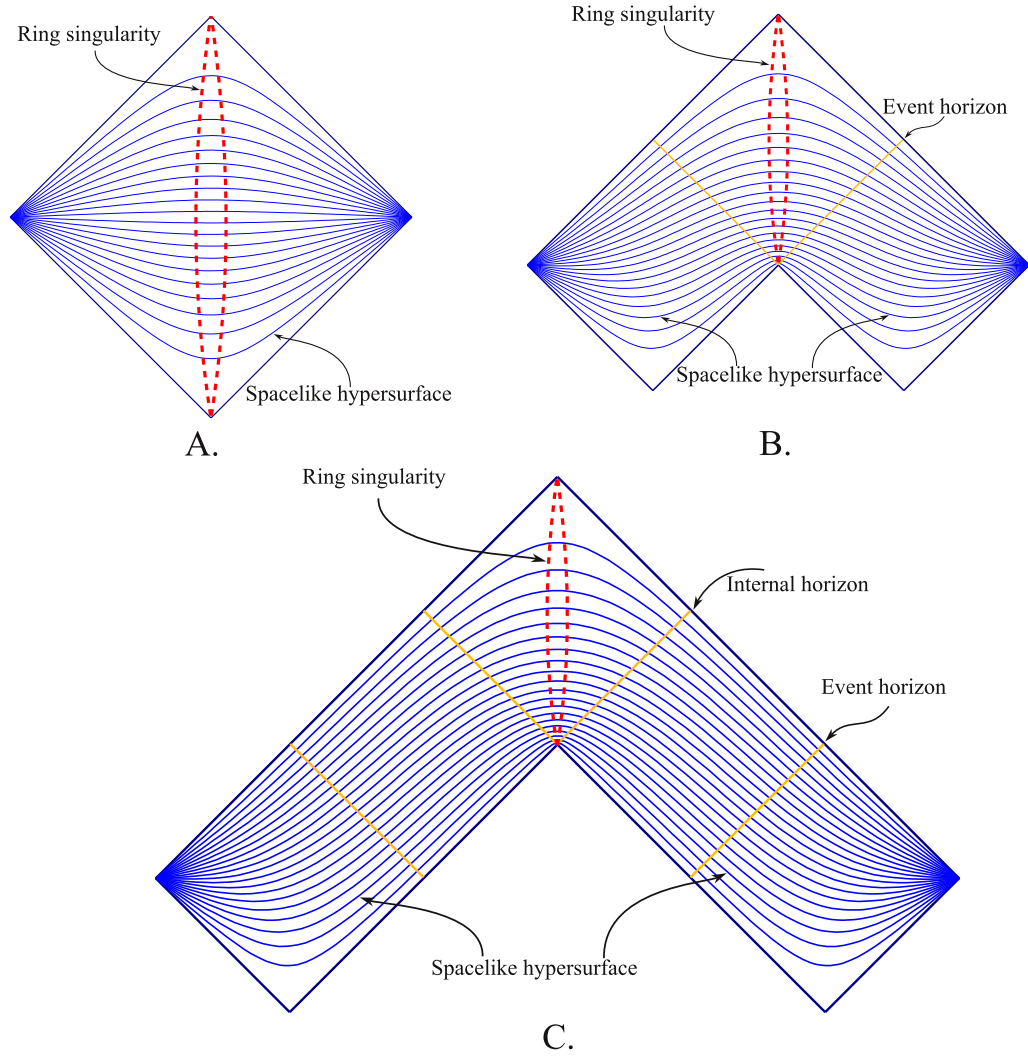


FIGURE 7.16: **A.** Space-like foliation of the naked Kerr-Newman solution ($a^2 + q^2 > m^2$). **B.** Space-like foliation of the extremal Kerr-Newman solution with $a^2 + q^2 = m^2$. **C.** Space-like foliation of the non-extremal Kerr-Newman solution ($a^2 + q^2 < m^2$).

7.3.5 The significance of the analytic extension at the singularity

The analytic extension beyond the singularity obtained here completes the series of results obtained for the Schwarzschild [101] and Reissner-Nordström [96] solutions. As in those simpler cases, it becomes clear that the singularity can coexist with the geometric and topological structures of the spacetime, in a way which doesn't destroy the information contained in the fields. As in the other cases, we can extrapolate for the case when the black hole is not eternal, *e.g.* when it evaporates. This is because the Kerr-Newman solution is, according to the no-hair theorem, representative for all kinds of black holes.

The fact that the metric is allowed to become degenerate is not a problem, because, as shown in [67, 73, 76], we have now the mathematical apparatus to deal with this kind

of singularities.

In conclusion, despite the singularities present inside the black holes, there is no reason to consider the Kerr-Newman black holes destroy causality, the evolution equations and the information conservation. The Kerr-Newman black holes are the most general stationary solution. The no-hair theorem makes them typical for our universe. They are typical even for the evaporating black holes, because the foliations presented here allow smooth modifications of the parameters m , q , and a , while preserving the topology. Moreover, we obtained charged singularities with smooth electromagnetic potential, leading to models of non-singular charged particles. This is why we can be more optimistic about the singularities of the general black holes as well.

This chapter is based on author's article [102].

7.4 Global hyperbolicity and black hole singularities

Black hole singularities are usually considered to break the time evolution, in (classical) General Relativity. It is shown here that, in fact, the standard black hole solutions are compatible with global hyperbolicity. For this, globally hyperbolic spacetimes containing singularities are constructed.

The first step was to extend solutions of the Einstein equation to the singularities, in a way which avoids infinities. The stationary black hole solutions are then analytically extended beyond the singularities. Next, the topology of the spacetime at the singularity is repaired, by using the new extensions. Then, solutions are constructed, which are shown to be globally hyperbolic by foliating the spacetime with Cauchy hypersurfaces. The foliations are constructed by applying a Schwarz-Christoffel mapping to the Penrose-Carter representations of the black hole solutions.

The results are applied to non-stationary black holes, including evaporating ones.

7.4.1 Introduction

7.4.1.1 The singularity theorems

Despite the successes of General Relativity, one of its own consequences seems to question it: the occurrence of singularities in the black holes. It is often said that General Relativity predicts, because of these singularities, its own breakdown [46, 48, 78–81].

Such singularities follow from the *singularity theorems* of Penrose and Hawking [5, 42–46]. The conditions leading to singularities were found to be common (Christodoulou [63]), and then even more common (Klainerman and Rodnianski [64]).

Initially, there was some confusion regarding the singularities. In 1916, when Schwarzschild proposed [133, 134] his solution to Einstein’s equation, representing a black hole, it was believed that the event horizon is singular. Only after 1924, when Eddington proposed another coordinate system which removed the singularity at the event horizon [65], and 1958, with the work of D. Finkelstein [66], it was understood that the event horizon’s singularity was only apparent, being due to the choice of the coordinate system. But the singularity at the center of the black hole remained independent of the particular coordinates, and the singularity theorems showed that any black hole would have such a singularity. In [96, 97, 101] we have shown that, although the genuine singularities cannot be removed, they can at least be made manageable – there are coordinate changes which make the metric degenerate, but smooth.

7.4.1.2 The black hole information paradox

Soon (in its proper time) after an object passes through the event horizon, it reaches the singularity of the black hole. All the information contained in it seems to vanish in the singularity.

On the other hand, the equations governing the physical laws are in general reversible, guaranteeing that no information can be lost. But according to Hawking [78, 82] the black hole may emit radiation and evaporate. If the black hole evaporates completely, it seems to leave behind no trace of the information it swallowed. Moreover, it seems to be possible for an originally pure state to end up being mixed, because the density matrix of the particles in the black hole’s exterior is obtained by tracing over the particles lost in the black hole with which they were entangled. This means that the unitarity appears to be violated, and the problem becomes even more acute.

7.4.1.3 The meaning of singularities

The singularities in General Relativity are places where the evolution equations cannot work, because the involved fields become infinite. If the Cauchy surface on which the fields are defined is affected by singularities, then the equations cannot be developed in time.

From geometric viewpoint, these singularities are points where the metric becomes singular, and the geodesics become *incomplete*. Since we don't know to extend the fields to such points, normally we remove them from the spacetime.

Actually, we can rewrite the fields involved, and the equations defining them, so that the fields remain finite at any point [40, 67, 96, 97, 101]. At the points where the metric is non-singular, the equations remain equivalent to Einstein's equation.

With this modification, the spacetime can be extended to the singular points.

Once we have the fields and the topology repaired, we have to check that we can choose a maximal globally hyperbolic spacetime (or equivalently, admitting a Cauchy foliation) so that the evolution equations can be defined.

Consequently, a natural interpretation of the singularities emerges, which makes them harmless for the physical law, in particular for the information conservation.

We will illustrate this approach on the black hole solutions known from the literature.

7.4.2 Canceling the singularities of the field equations

Some of the tensor fields involved in Einstein's field equation become infinite at the singularity points. We need a method to replace them with other fields which obey equations equivalent to Einstein's at the non-singular points, but remain in the same time smooth at the singularity. Our proposed approach generalizes the semi-Riemannian geometry, so that it works at an important class of singularity points [67, 73, 76].

Other method, used by Einstein and Rosen (for which they credited Mayer [40]) is to multiply the Einstein equations with by a well-chosen power of $\det g$. This way, all instances of g^{ab} in the expression of $\det g^2 \text{Ric}$ are replaced by $\det g g^{ab}$, the adjugate matrix of g_{ab} ([40], p. 74), and the singularities of the Riemann and the Ricci tensors can be canceled out for certain cases. The resulting equations still make sense as partial differential equations in the components of the metric tensor. Of course, the invariants of Semi-Riemannian Geometry have geometric meanings so long as the metric tensor is non-singular. But once we allow it to become singular, the invariants like covariant derivative and curvature become undefined. The quantities which replace them if we multiply the equations with $\det g$ or other factors cannot give them this meaning. Maybe a new kind of geometry is required to restore the ideas of covariant derivative and curvature for singular metrics.

We preferred to have a generalization of Semi-Riemannian Geometry, which allows us to define geometric objects like covariant derivative, Riemann curvature and Ricci tensor even when the metric becomes degenerate.

We did a modest step in this direction, by developing a Singular Semi-Riemannian Geometry for a class of metrics which can change the signature and become degenerate (chapter 2, and [67, 73, 76]³).

Our extension of Semi-Riemannian Geometry allows one to define in some cases covariant derivative, the Riemann tensor R_{abcd} (but not R^a_{bcd}), the Ricci tensor and the scalar curvature (hence the Einstein tensor). It can be used to write Einstein's equation, which in general becomes singular when the signature changes, but we can replace it with a densitized version which removes its singularities.

The metric of a stationary black hole solution like the Schwarzschild, Reissner-Nordström, and in general Kerr-Newman, has components which become infinite at singularity. Since the methods we developed in [67, 73, 76] work for metrics whose singular character manifests only by being degenerate, we may think that the stationary black hole solutions cannot be covered. But, in fact, we can make the singular metric be of degenerate type, by applying certain coordinate changes. We did this for the Schwarzschild metric in [101], for the Reissner-Nordström metric in [96], and for the Kerr-Newman metric in [97].

This approach is similar to the Eddington-Finkelstein coordinates, which were used to remove the apparent singularity on the event horizon of the Schwarzschild black hole [65, 66]. The new coordinates were chosen so that they are themselves singular at the (apparent) singularity, canceling it. In the case of the Eddington-Finkelstein coordinates, the resulting metric is non-degenerate on the event horizons, where previously appeared to be singular. In the case of the genuine singularities of the stationary black hole solutions, the metric obtained by the new coordinates we propose has analytic components, and its singularity reduces to the fact that it is degenerate.

In the following, we need to explore the arena on which are defined the fields – the spacetime – to make sure that the singularities don't alter the topology in a way which breaks down the evolution equations.

³Important results for the case when the metric has constant signature were been previously developed by D. Kupeli in [19, 20], by a different approach. Previous results were obtained by Moisi [6], Strubecker [7–10], and Vranceanu [11].

7.4.3 The topology of singularities

In the following we shall see that the main black hole solutions of Einstein's equations can be interpreted so that the time evolution is not jeopardized. Two main factors help us to do this:

1. Remove the singularities from the field equations.
2. Make sure that the time evolution takes place on a space whose topology doesn't change in time.

In the current section, we shall see that these conditions can be ensured for the typical black hole solutions of Einstein's equation.

7.4.3.1 The final space-like singularity from Schwarzschild's solution

The Schwarzschild solution has the particularity that its singularity is spacelike. Although in some coordinate systems (like the original Schwarzschild coordinates) the singularity is apparently timelike and one dimensional, it is in fact spacelike, as we can see by choosing Kruskal-Szekeres coordinates or Penrose-Carter coordinates.

Let's take the Penrose-Carter diagram of a Schwarzschild black hole (Figure 7.1).

This diagram actually represents the maximally extended Schwarzschild solution, in Penrose-Carter coordinates. This extended solution is interpreted to include, together with the universe containing the black hole, another universe, in which there is a white hole.

We can foliate this spacetime with Cauchy hypersurfaces, as we can see in Figure 7.17.

This foliation is obtained with the help of the version of the Schwarz-Christoffel mapping from equations (7.71) and (7.72) (*cf. e.g.* [128]).

To obtain the foliation from Figure 7.17, we take the prevertices to be

$$(-\infty, -a, a, +\infty, a+i, -a+i), \quad (7.98)$$

where $a > 0$ is a real number. The angles are respectively

$$\left(\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}\right). \quad (7.99)$$

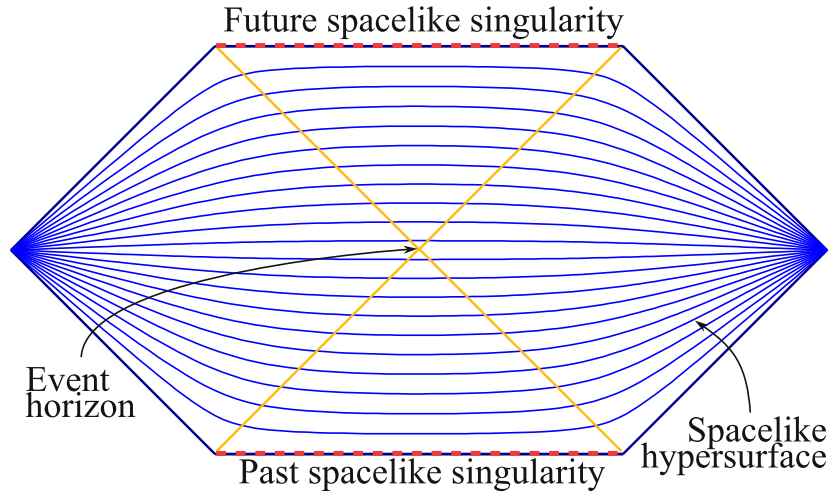


FIGURE 7.17: Space-like foliation of the maximally extended Schwarzschild solution.

This foliation contains the past white hole, which one may consider unphysical. We can make a similar foliation, this time without the white hole, if we use the prevertices

$$(-\infty, -a, 0, a, +\infty, b+i, -b+i), \quad (7.100)$$

where $0 < b < a$ are positive real numbers. The angles are respectively

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4} \right). \quad (7.101)$$

The resulted foliation is represented in Figure 7.18.

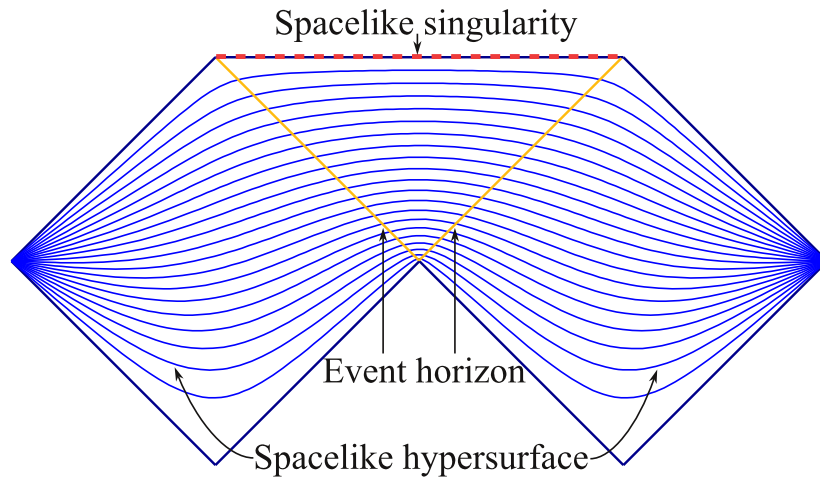


FIGURE 7.18: Space-like foliation of the Schwarzschild solution, without white hole.

7.4.3.2 Timelike singularities of charged black holes

The stationary solutions of Einstein's equations representing charged black holes were discovered by Reissner [125] and Nordström [126]. The Penrose-Carter diagrams of the maximally extended solutions are represented in Figure 7.7 A for naked black holes, 7.7 B for extremal black holes, and 7.7 C for non-extremal black holes.

In section §7.2.5 we proposed new coordinates for the Reissner-Nordström metric, which make the metric smooth at the singularity [96]. The singularity appears to be on a timelike line, with respect to which the solution is symmetric. We take as prevertices of the Schwarz-Christoffel mapping (7.72) the set from equation (7.75). By an appropriate choices of a we get the foliation represented in diagrams 7.11, 7.13, respectively 7.12, corresponding to the Figures 7.7 A, 7.7 B, respectively 7.7 C.

These solutions ignore the other universes appearing when the equations are analytically extended to their maximum domain in Penrose-Carter coordinates. It is debatable if we really need, from physical viewpoint, to include extensions beyond the Cauchy horizons, and by them the infinite number of other universes appearing in such maximal extensions. Our foliation takes the outside and the inside of the black hole and foliates it as required to solve this problem. Taking the maximally extended solution would not allow us to do this, because it contains Cauchy horizons, beyond which the evolution equations cannot be evolved causally ([5], p. 159, [117], p. 317).

Also, we shall see that, by limiting to one universe in these solutions, instead of taking the maximal extensions, we can combine them with non-singular solutions and have black holes with a beginning and an end. Such black holes seem more plausible from physical viewpoint, although we will not exclude the primordial and eternal black holes.

7.4.3.3 Timelike singularities of rotating black holes

The rotating neutral black holes (Kerr, [131]), and the rotating electrically charged black holes (Kerr-Newman, [129]) don't have spherical symmetry, they are axisymmetric. Their Penrose-Carter diagrams are similar with those for Reissner-Nordström black holes (Figure 7.7), in coordinates (t, r) , but there are some major differences. In Boyer-Lindquist coordinates the singularity appears to be a ring, and in its neighborhood the coordinate φ becomes timelike – the curves defined by $(t = \text{const.}, r = \text{const.}, \vartheta = \text{const.})$ are closed timelike curves (*cf. e.g.* [5], §5.6, [117], p. 315)⁴. In [97] we proposed new coordinates, which make the metric analytic at the singularities, and remove, as a

⁴It has been proposed that the closed time-like curves depend on the particular coordinate system used [130].

byproduct, the closed timelike curves. These new extensions of the metric can be taken so that the solution looks more like the Reissner-Nordström solution. So, the Penrose-Carter diagrams for our extensions of the Kerr and Kerr-Newman solutions look very similar to those for the Reissner-Nordström solution (Fig. 7.16).

7.4.4 Non-stationary black holes

The stationary solutions are idealizations. In reality, a black hole doesn't necessarily exist from the beginning of the universe, it may be younger. Also, it may evaporate completely after a finite time interval.

For example, a spacelike foliation of a non-rotating and electrically neutral black hole which formed after the beginning of the Universe, and which continues to exist forever, is represented in Figure 7.19.

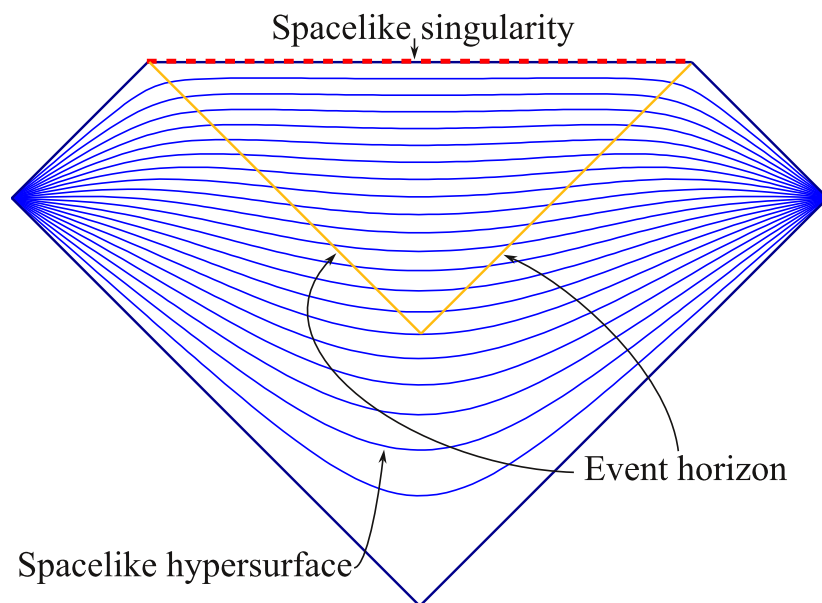


FIGURE 7.19: The spacelike foliation of a non-rotating and electrically neutral black hole formed after the beginning of the Universe, which continues to exist forever.

The prevertices of the Schwarz-Christoffel mapping (7.72) whose image is represented in Figure 7.19 are given by the set

$$(-\infty, 0, +\infty, a + i, -a + i), \quad (7.102)$$

where $0 < a$ is a positive real number. The angles are respectively

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}\right). \quad (7.103)$$

We see that, because the typical spacelike hypersurface in the foliation in Figure 7.18 is diffeomorphic with the space \mathbb{R}^3 of the Minkowski spacetime $\mathbb{R}^3 \times \mathbb{R}$, the topology doesn't change because of the occurrence of a neutral non-rotating black hole. In fact, the typical spacelike hypersurface of the foliation of a charged and rotating black hole also has the same topology as \mathbb{R}^3 , as we have seen. So they can appear and evaporate as well in an \mathbb{R}^3 space, without disrupting the topology. This condition is required to have a good time evolution.

If the non-rotating and electrically neutral black hole is primordial (exists from the beginning of the universe), but evaporates completely after a finite time, the spacelike foliation is as represented in Figure 7.20. The prevertices and the angles are identical to those for Figure 7.13.

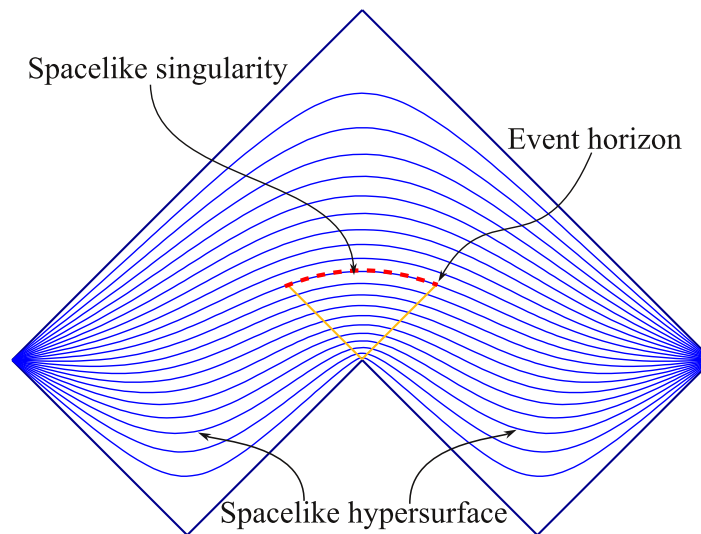


FIGURE 7.20: The spacelike foliation for a non-rotating and electrically neutral primordial black hole, which evaporates after a finite time.

If this non-rotating and electrically neutral black hole is not primordial and it evaporates completely after a finite time, then at large distances the spacetime is very close to the Minkowski spacetime, being asymptotically flat. Consequently, a spacelike foliation looks like that in Figure 7.21.

The prevertices of the diagram represented in Figure 7.21 are the same as equations (7.73) and (7.74).

Let's see now what happens if the singularity is timelike, as in the case of the charged and/or rotating black holes. If the black hole is primordial and evaporates, the corresponding spacelike foliation is represented in Figure 7.22.

The prevertices and the angles are the same as those for the Figure 7.13.

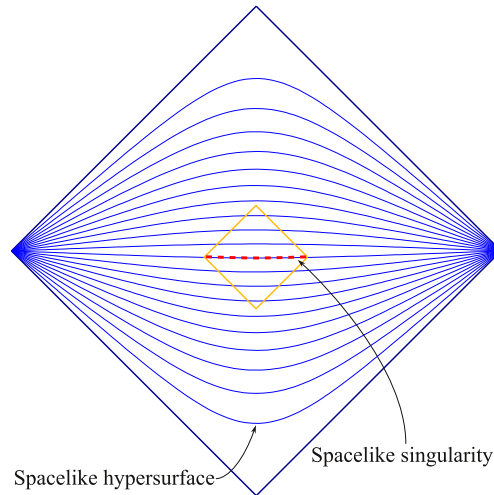


FIGURE 7.21: The spacelike foliation for a non-rotating and electrically neutral black hole formed after the beginning of the Universe, and which evaporates after a finite time.

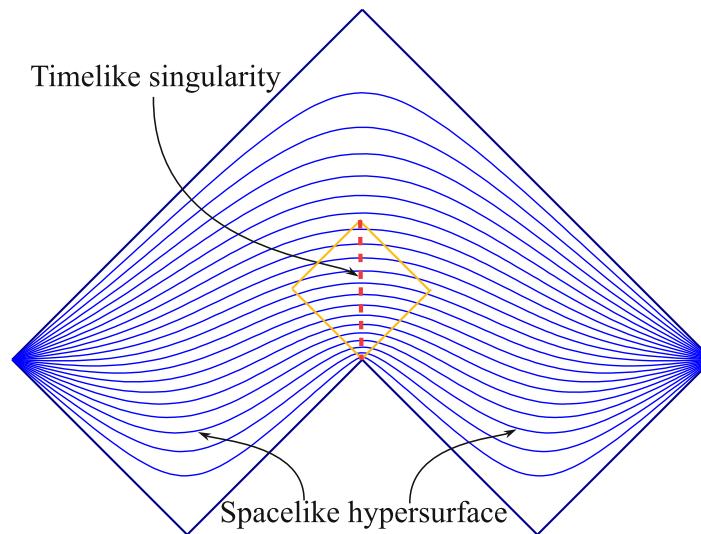


FIGURE 7.22: Primordial evaporating black hole with timelike singularity.

The spacelike foliation of a spacetime containing a black hole which is not primordial and does not evaporate is represented in Figure 7.23. The prevertices and the angles are again those from equations (7.73) and (7.74).

The same prevertices and angles are used to construct the spacelike foliation for a non-primordial evaporating black hole, represented in Figure 7.14. If the black hole rotates, the singularity is ring-shaped (as in Fig. 7.16), but the diagrams are similar.

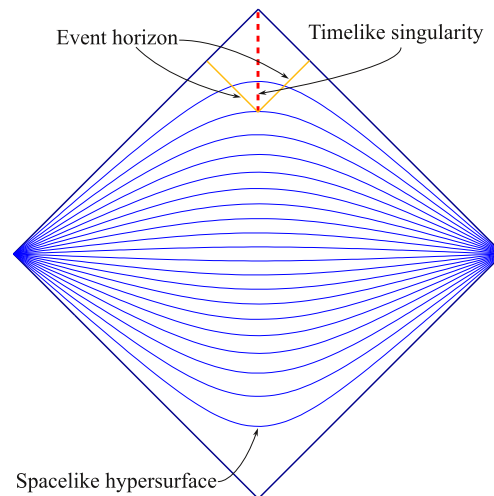


FIGURE 7.23: Non-primordial non-evaporating black hole with timelike singularity.

7.4.5 Time evolution in spacetimes with singularities

The conformal structure is given equivalently by the lightcones. This structure is involved in the Cauchy development in General Relativity. Namely, each point in spacetime is affected by what happens in its past lightcone.

In our formulation the singularities are repaired by a coordinate change, and the topology of the spacelike hypersurface is preserved by the time evolution. The conformal structure is identical, in all cases examined, to the conformal structure of non-singular spacetimes which are globally hyperbolic. Namely, their Penrose-Carter diagrams are all conformally equivalent to that of the Minkowski spacetime. This is evident from the conformal equivalence of all solutions with the strip \mathcal{S} from (7.71), via a Schwarz-Christoffel mapping of the form (7.72). Therefore, the property of global hyperbolicity of the Minkowski spacetime transfers to our solutions as well.

This analysis shows that, even in the presence of singularities, the physical laws don't necessarily break down. We can make

1. an appropriate modification of the equations so that the metric becomes smooth at the singularities (see §7.4.2)
2. an appropriate foliation of the spacetime into spacelike hypersurfaces (*cf.* §7.4.3 and §7.4.4)
3. an appropriate extension of the spacetime at singularities, so that the topology of the spacelike hypersurfaces of the foliations is preserved (*cf.* §7.4.3 and §7.4.4).

These choices, which are not determined by the non-singular General Relativity, ensure us that the time evolution is not disrupted, and the Cauchy data, hence the information, is preserved. Also, this allows the unitarity to be restored, because there is no loss in the information, and a pure state can no longer become mixed by this mechanism.

7.4.6 Singular General Relativity

There are many actions we can take when encountering singularities in the solutions of Einstein's equation. There is no need to *cancel* them [60], or to say about them “here be dragons” – we just have to explore them theoretically.

This chapter presents an approach to the problems posed by the singularities.

One way to repair the singularities in the Einstein equation is by using the idea proposed by Einstein and Rosen [40] (see §7.4.2).

But we can do even more. Einstein's equation is an identity between Einstein's tensor, which is defined by the curvature, and the stress-energy tensor of the matter fields. By multiplying them with some functions, so that the partial differential equations expressed in a coordinate system become non-singular, we can indeed extend them at the singularities. But the quantities involved now seem to have no meaning, unlike the original terms.

This suggests that it may be useful to develop the Semi-Riemannian Geometry to deal with such situations when the metric is degenerate or singular.

We already explored the case when the metric is allowed to pass from being non-degenerate to being degenerate in [67, 73, 76]. We showed there that we can still have invariants like the familiar Riemann curvature and like the covariant derivative, if the metric satisfies some reasonable conditions. This allowed us to write a densitized version of Einstein's equation, which remains smooth at the singularities due to the degeneracy of the metric, although the curvature operator R^a_{bcd} , and the Kretschmann scalar $R_{abcd}R^{abcd}$ become singular. Normally, the Einstein tensor $R_{ab} - \frac{1}{2}g^{ab}R_{ab}$ also becomes singular, in which case we can work with its densitized version. But there are cases when it remains smooth, for example in the vacuum solutions like Schwarzschild's and Kerr's (when it is actually 0).

7.4.7 Conclusions

This chapter shows how one can extend smoothly the black hole solutions at the singularities, then find an appropriate foliation of the spacetime into spacelike hypersurfaces,

and an appropriate extension of the spacetime at singularities, so that the topology of the spacelike hypersurfaces of the foliations is preserved. In our typical examples, the time evolution in the presence of singularities is restored. The examples given here showed this explicitly for the neutral and charged, rotating or non-rotating, primordial or not, evaporating or not black holes. These models have been shown here to have Penrose-Carter diagrams conformally equivalent to that of the Minkowski spacetime, inheriting therefore from the latter the global hyperbolicity. Consequently, the Cauchy data is preserved, and the information loss is avoided. This allows the construction of Quantum Field Theories in such curved spacetimes ([79], p. 9), and the unitarity is restored.

Chapter 8

Quantum gravity from metric dimensional reduction at singularities

This chapter contains parts of author's article [118]. A series of old and recent theoretical observations suggests that the quantization of gravity would be feasible, and some problems of Quantum Field Theory would go away if, somehow, the spacetime would undergo a dimensional reduction at high energy scales. But an identification of the cause of this dimensional reduction would still be desirable.

A possible explanation of the dimensional reduction is suggested by recent results in understanding the geometry of singularities in General Relativity. These new methods don't require modification of General Relativity, being just extensions of its mathematics to the limit cases. They turn out to work fine for some known types of cosmological singularities (black holes and FLRW Big-Bang), allowing a choice of the fundamental geometric invariants and physical quantities which remain regular. The resulting equations are equivalent to the standard ones outside the singularities.

One consequence of this mathematical approach to the singularities in General Relativity is a special, (geo)metric type of dimensional reduction: at singularities, the metric tensor becomes degenerate in certain spacetime directions, and some properties of the fields become independent of those directions. Effectively, it is like one or more dimensions of spacetime just vanish at singularities. Therefore, it seems that the geometry of singularities leads naturally to the spontaneous dimensional reduction needed by Quantum Gravity.

8.1 Introduction

Quantum Field Theory (QFT) and *General Relativity* (GR) are the most successful theories in fundamental theoretical physics. Their predictions were confirmed with very high precision, and they seem to offer accurate and complementary descriptions of the physical reality.

Yet, each one of them has some problems, especially when one tries to combine them. GR has the problems of infinities which appear at the *singularities*. QFT also has problems with infinities, which appear in the *perturbative expansion*, and are usually approached by renormalization techniques. Fortunately, the *renormalization group* formalized and solved many of the problems of QFT [135–140]. The *Standard Model* of particle physics is proven to be renormalizable [141–143], although the solution depends crucially on the existence of the Higgs boson.

But arguably the greatest difficulties appear when one tries to quantize gravity. General Relativity without matter fields is perturbatively non-renormalizable at two loops [120, 121]. It requires an infinite number of higher derivative counterterms with their coupling constants. The main reason is the dimension of Newton’s constant, which is $[\mathcal{G}_N] = 2 - D = -2$ in mass units.

In the quest of understanding the *small scale, ultraviolet* (UV) limit in QFT, and especially in the approaches to Quantum Gravity (QG), is accumulated evidence which seems to point in one particular direction. This evidence suggests, or even requires, that there is a *dimensional reduction* to two dimensions in the UV limit.

While many distinct approaches agree that somehow a dimensional reduction will solve the main problems of the interface between QFT and GR, what seems to be missing is the explicit cause leading to this spontaneous reduction.

As we will see in this chapter, the dimensional reduction is ensured by the spacetime geometry at singularities in a very concrete way.

Usually, the apparent incompatibility between QFT and GR which manifests as non-renormalizability is considered to be the fault of the latter, hence usually the unification proposals start by modifying GR. Various approaches to QG are viewed as a hope which will cure not only the non-renormalizability, but also the problem of singularities, with the price of giving up one or more fundamental principles of GR.

Here we will take the opposite position: the solution to the problem of singularities comes from GR, and it also leads to the desired two-dimensionality in the UV limit, which is needed by quantum gravity.

The following can be considered a definition of Quantum Gravity (*cf.* 't Hooft [144]):

Quantum Gravity is usually thought of as a theory, under construction, where the postulates of quantum mechanics are to be reconciled with those of general relativity, without allowing for any compromise in either of the two.

Here we will try to see how far we can go in reconciling QFT and GR without making any compromises.

In this chapter, we aim to show that the solution to the problems of singularities in GR, developed in [67, 73, 84, 96–102] and reviewed briefly in §8.3, has implications to Quantum Gravity. Various approaches to QG suggest that if the spacetime becomes 2-dimensional at small scales, the quantization of gravity will become possible. Some of these hints will be reviewed in section §8.2. While dimensional reduction appears to be a desirable ingredient for QG, it would be useful to have an explanation of the reason which lead to the dimensional reduction, and a geometric interpretation of its meaning. In section §8.4, we will explain how the benign singularities cause the number of dimensions to be reduced, because of the way the metric becomes degenerate. Then we will try to connect the properties of the dimensional reduction caused by singularities, to those required by some of the approaches to QG.

8.2 Hints of dimensional reduction coming from other approaches

The method of regularization through dimensional reduction appeared from the observations that the loop integrations depend on the dimension in a continuous way, so that we can replace the dimension 4 by $4 - \varepsilon$, avoiding the poles, and at the end make $\varepsilon \rightarrow 0$ [141, 142, 145]. The original method of dimensional regularization is rather formal, and apparently without implications to the actual physical dimensions. On the other hand, the fact that Quantum Gravity works fine in two dimensions justifies the consideration of the possibility that at small scales the number of dimensions is indeed reduced. In the following we review some “signs” that the spacetime is actually required to become two-dimensional in the small scale limit, while maintaining four dimensions at large scales.

8.2.1 Dimensional reduction in Quantum Field Theory

Suggestions that the two-dimensionality plays an important role appeared in various contexts of QFT. Since the first exactly solvable QFT model was discovered [146], the two-dimensional QFT proved to lead in a non-perturbative and direct manner to interesting results which can be applied then to make conjectures and find results for four-dimensions (see [147] and [148] and references therein).

8.2.1.1 Two-dimensional QCD

The two-dimensional theories are not necessarily just toy models. There are strong suggestions that the scattering amplitudes in QCD can be obtained from a two-dimensional field theory [149–151], and models in which “this two-dimensional nature of the interactions is manifest” appeared in the context of high energy Regge regime [152, 153].

8.2.1.2 Fractal universe and measure dimensional reduction

The *fractal universe program* developed by G. Calcagni originated from the idea of keeping from the Hořava-Lifschitz gravity the feature that it leads to a two-dimensional phase in the UV regime, but in the same time aims to remain Lorentz invariant. For this, it maintains an isotropic scaling, but to compensate it replaces the standard measure used in the action with a measure (initially a Lebesgue-Stieltjes measure) which reduces the Hausdorff dimension to two at the ultraviolet fixed point. The action becomes fractional, and the resulting theory is dissipative, hence non-unitary, although the energy turns out to be conserved.

The action is taken to be of the form

$$S = \int_{\mathcal{M}} d\varrho(x) \mathcal{L}(\phi, \partial_\mu \phi) \quad (8.1)$$

with a measure

$$d\varrho(x) = v dx^D \quad (8.2)$$

Initially it was explored the theory with weight of the form $v := \prod_{\mu=0}^{D-1} f_{(\mu)}(x)$. Taking all functions $f_{(\mu)} = f$ leads to an isotropic measure, but one can make also anisotropic choices.

As an example, the scalar field is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \quad (8.3)$$

where f is taken to have the dimension $[f] = 1 - \alpha$, so that $[v] = D(1 - \alpha)$.

The scaling dimension of the scalar field ϕ is

$$[\phi] = \frac{D\alpha - 2}{2} \quad (8.4)$$

and vanishes if and only if $\alpha = \frac{2}{D}$, or $\alpha = \frac{1}{2}$ for $D = 4$.

If the potential is polynomial in ϕ

$$V = \sum_{n=0}^N \sigma_n \phi^n, \quad (8.5)$$

the resulting engineering dimension is

$$[\sigma_N] = D\alpha - \frac{N(D\alpha - 2)}{2} \quad (8.6)$$

and to require the theory to be power-counting renormalizable, one imposes that N is restricted by the condition $[\sigma_N] \geq 0$. For $\alpha = \frac{D}{2}$, N is unconstrained, while for $D = 4$ and $\alpha = 1$, one can choose $N = 4$. The idea is to construct a $D = 4$ theory so that in the infrared limit $\alpha_{IR} = 1$ and $N = 4$, and in the ultraviolet limit, $\alpha_{UV} = \frac{1}{2}$.

Similar principles lead to a modified action for General Relativity, from which a modified version of Einstein's equation is derived.

The theory introduced in [154–156] was further refined and put on more rigorous mathematical basis in [157–163].

Mathematically, the fractal universe theory relied initially on the Lebesgue-Stieltjes measure (subsequently replaced by fractional measures [158]), fractional calculus, and fractional action principles [164–166].

In section §8.4.7 we will see that a measure of the form (8.2), having the desired properties, emerges naturally from our theory of singularities.

8.2.1.3 Topological dimensional reduction

The spacetime is considered, in general, to be a D -dimensional *topological manifold* – that is, its topology is locally like that of \mathbb{R}^D , and usually $D = 4$. What would be the implications of a *topological dimensional reduction*, that is, if D would have, at some regions, a lower topological dimension?

In [167], D.V. Shirkov studied a $g\varphi^4$ QFT model described by a self-interacting Lagrangian $L = T - V$, where

$$V(m, g; \varphi) = \frac{m^2}{2}\varphi^2 + \frac{4\pi^{d/2}M^{4+d}}{9}g_d\varphi^4, \quad g > 0 \quad (8.7)$$

This theory was initially motivated by the possible non-existence of a Higgs boson of 140 ± 25 , and aimed to explore the Ginzburg-Landau-Higgs alternative of a constant classical Higgs field at ~ 250 GeV. But the absence of a Higgs quantum field makes the weak force again non-renormalizable. To obtain the regularization, D.V. Shirkov worked in a spacetime with variable topology, having a number of dimensions which varies from $D = 4$ in the IR limit, to $D = 1 + d < 4$ in the UV limit. The coupling constant was assumed to run from dimension $D = 4$ in the IR regime, to $D = 2$ in the UV regime.

One fundamental principle used was the *DR Agreement*, stipulating an equivalence between the reduction at spacetime scale $x_{dr} \sim \frac{1}{M_{dr}}$ and the reduction at the energy-momentum scale $p_{dr} \sim M_{dr}$.

The idea is illustrated by the approximation with a manifold obtained by joining two cylinders $S_{R,L}$ and $S_{r,l}$, of radii $R > r$ and lengths L, l , with a transition region S_{coll} of varying radius. Then, one can think at various problems taking place on the obtained manifold, for example solve some equations, and then take the limit $r \rightarrow 0$.

The replacement of the volume element in the momentum space

$$d^4k \rightarrow d_M k = \frac{d^4k}{1 + \frac{k^2}{M^2}} \quad (8.8)$$

yields a one-loop Feynman integral which has as the IR asymptote the function

$$\ln \frac{q^2}{m_i^2}, \quad (8.9)$$

and as the UV asymptote the function

$$\ln \frac{4M^2}{m_i^2} + \frac{M^2}{q^2} \ln \frac{q^2}{M^2}. \quad (8.10)$$

This results in a reversal of the running coupling evolution in the two-dimensional region, and the value of the coupling constant gains a finite minimum value $\bar{g}_2(\infty) < \bar{g}_2(M_{dr}^2)$, where M_{dr} is the reduction scale. The coupling constant decreases in the IR limit as expected, but it has a maximum at $q = M_{dr}$, and in the UV limit decreases again, to the minimum $\bar{g}_2(\infty)$.

An interesting possibility which occurs is a novel type of *Grand Unified Theory* scenario, where the coupling constants of the Standard Model forces converge without needing an $SU(5)$ or other leptoquark symmetry¹. As stated in the concluding section of [167]:

The notable observation is that the change of geometry could yield the same final result as an explicit change of dynamics (by adding leptoquark fields etc.).

Of course, in this case the unification scale seems to require a bit higher energies than the GUT unification scale, taking place in the two-dimensional region, which is difficult to probe:

Among further quests that are in order, let us put in the first place the issue of examining the chance of detecting some physical signal “through the looking-glass at scale M ” that would provide us with direct evidence on the existence of dimension reduction of any kind.

The idea of topological dimensional reduction is further explored by P. Fiziev and D. V. Shirkov in [124, 170–172], where it is applied to Klein-Gordon equation. In [170], the Klein-Gordon equation is studied on a spacetime which is the direct product between a 2-dimensional surface of revolution with variable radius and topology, and the 1-dimensional manifold $(\mathbb{R}^1, -dt^2)$ representing the time dimension. The dimensional reduction takes place between $1 + 2$ and $1 + 1$ dimensions, by reducing the radius of revolution to 0.

By the method of separation of variables, the KG equation is reduced to three ordinary differential equations, from which two are simple wave equations. The third one is more difficult, but it can be put by a change of variables in the form of a one-dimensional Schrödinger-like equation, with the potential V defined by the geometry. The obtained solution forbids the propagation of signals related to the physical degrees of freedom related exclusively to the higher dimension, into the lower dimension region.

The resulting spectra depend on the mass from the KG equation and on the shape of the transition between the two regions of constant radii, and correspond to possible particles and their antiparticles. This suggests the idea that [170]

The specific spectrum of scalar excitations resembles the spectrum of the real particles; it reflects the geometry of the transition region and represents its “fingerprints”

¹For introductions to various GUT scenarios see [168, 169].

The possibility of deducing the geometry of the transition region by the “fingerprints” from the corresponding spectrum suggests an answer to the question about possible experimental “evidence on the existence of dimension reduction of any kind”, asked in the first paper [167].

Some of these results were further generalized to higher dimension and multiple variable radii of compactification [171]. The manifold under consideration is a hypersurface in a semi-Euclidean $2d$ -dimensional space \mathbb{R}_{2d-1}^{2d} (see [3] for the notation), described by

$$\begin{cases} x^0 = t, & \dots, & x^{2k-1} = \rho_k(z) \cos(\phi_k), & \dots, \\ x^{2d-1} = z, & \dots, & x^{2k} = \rho_k(z) \sin(\phi_k), & \dots \end{cases} \quad (8.11)$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, $\phi_k \in [0, 2\pi]$, $\rho_k(z)$ are the shape functions (or the radii of compactification), and $k \in \{1, 2, \dots, d-1\}$. This hypersurface is a Lorentzian manifold of dimension $1 + d$. The solution turned out also in this general case to be reducible to one-dimensional Schrödinger-like equations, and to be regular even for $\rho_k = 0$.

The next step was to generalize the $(1+2)$ -dimensional solution to the non-static case of *axial universes*, by allowing the shape function to depend also on time, $\rho = \rho(t, z) \geq 0$ [124]. A $(1+2)$ -axial universe is a semi-Riemannian manifold which is a hypersurface of the Minkowski spacetime \mathbb{R}_3^4 , defined by

$$\begin{cases} x^0 = t, & x^1 = \rho(t, z) \cos(\phi), \\ x^3 = z, & x^2 = \rho(t, z) \sin(\phi) \end{cases} \quad (8.12)$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $\phi \in [0, 2\pi]$.

The Einstein Equation translated into three equations connecting ρ and its partial derivatives with the energy-momentum tensor. Various energy-momentum tensors were considered in the Einstein Equation in $1+2$ dimensions:

1. in vacuum, where $\rho(t, z)$ was obtained to be linear in the variables t, z , respectively both of them,
2. with positive cosmological constant Λ , resulting in a static spherical universe,
3. in the presence of standard “dust”, resulting in a homogeneous Monge-Ampère equation for ρ .

The solutions turned out to be integrable. The obtained dimension reduction points admit a classification, and there are hopes that they will provide new insights into the nature of the violations of the C, P, and T symmetries [170]:

The parity violation due to the asymmetric character of the construction of our models could be related to violation of the CP symmetry.

Then, the Klein-Gordon equation was solved on the resulting spacetimes.

Section §8.4.2 establishes a strong connection between these results in topological dimensional reduction and our approach.

8.2.1.4 Vanishing dimensions

One problem concerning the UV divergences in the Standard Model is the need to fine-tune the mass of the Higgs boson to an accuracy of 10^{-32} , to prevent the destabilization of the electroweak symmetry breaking scale [173]. Since fixing the cutoff at the electroweak scale makes the SM work well, it is believed that this is an indication of new physics beyond this scale. In [173] it is explored the alternative of keeping unmodified the structure of the SM, and reduce the dimension. One central point is that reducing the number of space dimensions to $d = 2$ makes the Higgs terms linearly divergent, and $d = 1$ makes them logarithmically divergent, making unnecessary the fine-tuning of the Higgs mass.

8.2.2 Dimensional reduction in Quantum Gravity

A review of the hints that, in various approaches to QG, a dimensional reduction occurs at small scales, is done by Carlip [174, 175]. Many of these hints involve the *spectral dimension*. The spectral dimension was calculated for causal dynamical triangulations (CDT) in [176], as evidence showing that the four-dimensional spacetime is recovered at larger scale. This resulted in a trend that various approaches to Quantum Gravity adhered to, consisting in calculating the spectral dimension in the UV limit to see if it is 2 [175, 177–179]. As it is known (see *e.g.* [124, 163, 180, 181]), while there is a correlation between the spectral dimension and the spacetime dimension, they are not equivalent. While the spectral dimension depends of the spacetime geometry too, it is very different. It represents the effective dimension of the diffusion process, being related to the dispersion relation of the corresponding differential operator. Spectral dimension is a widely used indicator in quantum geometry, and even if it is not equivalent with topological dimension, the latter bounds it.

As pointed out in [122], in dimension lower than four the Weyl curvature vanishes, and the vacuum Einstein equation has only locally flat solutions, or of constant curvature if the cosmological constant is not 0. This leads to the absence of local degrees of freedom,

i.e. of gravitational waves for the Classical Gravity, and of gravitons for QG. Our own approach leads in a surprising way to this kind of dimensional reduction and vanishing of the Weyl curvature §8.4.3.

8.2.2.1 The renormalization group: asymptotic safety

General Relativity appears to be non-renormalizable, but the renormalization group analysis may give us useful hints. One possibility is that the infinite number of coupling constants become “unified” when approaching the UV fixed point. In [182] S. Weinberg proposed, as solution to the non-renormalizability of Quantum Gravity, the idea of *asymptotic safety*. The number of coupling constants needed by an asymptotically safe Quantum Gravity theory in four-dimensional spacetime is infinite, but their values are required to remain finite and to converge, under the renormalization group flow, to an ultraviolet fixed point. Also, the *ultraviolet critical surface* of the renormalization group flow is required to be finite-dimensional. In order to make the $3+1$ -dimensional gravity asymptotically safe, the Einstein-Hilbert Lagrangian density $\mathcal{L} = -\frac{1}{16\pi\mathcal{G}}R\sqrt{-\det g}$ has to be supplemented with higher-order curvature terms, for example by modifying it to $\mathcal{L} = f(R)\sqrt{-\det g}$. As stated in [182], “there is an asymptotically safe theory of pure gravity in $2+\varepsilon$ dimensions, with a one-dimensional critical surface”, and “[a]symptotic safety is also preserved when we add matter fields”, provided we add certain compensatory fields.

Some evidence accumulated in favor of asymptotic safety [183–188], especially near two dimensions [189, 190]. The spectral dimension near the fixed point appears to be $d_S = 2$ [177]. In [185] is showed that the existence of a non-Gaussian fixed point for the dimensionless coupling constant $g_N = \mathcal{G}_N\mu^{d-2}$ requires two-dimensionality.

8.2.2.2 Causal dynamical triangulations

In the *causal dynamical triangulations* approach [176, 191–194], spacetime is approximated by flat four-simplicial manifolds, similar to quantum Regge calculus [195]. The spacelike edges are taken to be of equal length, and the timelike edges of equal duration. The causality is enforced by requiring a fixed time-slicing at discrete times, and that the time-like edges agree in direction. The path integrals can be calculated non-perturbatively, resulting in four-dimensional spacetimes. The spectral dimension, which is the dimension as seen by a diffusion process, turns out to be four at large distances, but two at short distances [176].

8.2.2.3 Hořava-Lifschitz gravity

Inspired by the quantum critical phenomena in condensed matter systems, Hořava proposed in 2009 a model of Quantum Gravity [196]. The starting assumption is that the space and time behave differently at scaling – there is an anisotropic scaling invariance:

$$\begin{cases} \mathbf{x} & \mapsto b\mathbf{x}, \\ t & \mapsto b^z t. \end{cases} \quad (8.13)$$

To describe an UV fixed point, the critical exponent turns out to be $z = D - 1 = 3$, although it is argued that $z = 4$ would be even better. This anisotropy is not required to be a symmetry of the action itself, but of the solutions. The theory describes in the UV limit interacting non-relativistic gravitons, and is power-counting renormalizable in $1 + 3$ dimensions.

Lorentz invariance is absent in the UV limit, but it is conjectured that it emerges at large distances, where it is hoped that $z \rightarrow 1$. The resulting field equations are second-order in time, to avoid ghosts. In the same time they are of high order in space, canceling the divergences of the loop integrals. The spectral dimension turns out to be again two, for high energies, and four for low energies [178].

The anisotropy breaks the diffeomorphism invariance, and picks out a distinct time direction. This can be expressed in a space+time foliation \mathcal{F} , as in the ADM formalism [41]. The group of diffeomorphisms $\text{Diff}(M)$ reduces to that of diffeomorphisms which preserve the leaves in the foliation, $\text{Diff}_{\mathcal{F}}(M)$.

If the lapse function N depends on the time only, it is called *projectable*, otherwise it is called *non-projectable*. Theories with projectable N were the first to be studied, but now the hope moved toward the non-projectable ones.

The action splits naturally into a kinetic term \mathcal{S}_K and a potential term \mathcal{S}_V :

$$\mathcal{S} = \mathcal{S}_K - \mathcal{S}_V. \quad (8.14)$$

The kinetic term \mathcal{S}_K of the action is defined as the most general invariant term built from at most two time derivatives of the metric. In terms of the *extrinsic curvature*

$$K_{ij} := \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \quad (8.15)$$

it has the form

$$\mathcal{S}_K = \frac{2}{\kappa^2} \int (K^{ij} K_{ij} - \lambda (K^s_s)^2) \sqrt{\det^3 g} N dt d^3x \quad (8.16)$$

where

$$\frac{2}{\kappa^2} = \frac{1}{16\pi\mathcal{G}_N}. \quad (8.17)$$

The dimension of the volume element is

$$[dt d^d \mathbf{x}] = -d - z \quad (8.18)$$

and each time derivative increases the dimension by

$$[\partial t] = z \quad (8.19)$$

therefore, the scaling dimension of κ is

$$[\kappa] = \frac{z - d}{2}, \quad (8.20)$$

so κ is dimensionless when $z = d$.

The potential term \mathcal{S}_V includes terms up to six spatial derivatives of the metric.

The standard ADM action is

$$\mathcal{S}_{EH} = \frac{1}{16\pi\mathcal{G}_N} \int (K^{ij} K_{ij} - (K^s_s)^2 + {}^3R - 2\Lambda) \sqrt{\det {}^3g} N dt d^3x \quad (8.21)$$

with the kinetic term $\lambda = 1$, which was hoped to emerge at large scale.

Some possible inconsistencies, internal and with the observations, in particular concerning the strong coupling and violations of unitarity, are discussed in [197–206]. Many of these objections arise from the difficulty to prove that GR is recovered in the IR limit.

A connection between our approach and Hořava-Lifschitz gravity is explored in section §8.4.6.

8.2.2.4 Hints from other approaches

As pointed out in [174, 175], there are hints from high temperature string theory [207] that the thermodynamic behavior becomes two-dimensional at high temperatures. Also, Modesto argued that in Loop Quantum Gravity the effective spectral dimension varies from four at large scales, to two at small scales [179, 208, 209]. Other results concern the spectral dimension in quantum spacetime based on noncommutative geometry [210, 211] and un-gravity [212].

8.3 Singularities in General Relativity

The other problem which seems to plague General Relativity is that of singularities. Under general conditions, the evolution equations in GR lead to singularities [5, 42–46]. It seems that they are unavoidable. The options seem to be:

1. give up GR, or at least modify it, and
2. explore the singularities and try to find alternative but equivalent descriptions, which don't have problems with the infinities.

Since the first approach has been widely explored in the literature, we focused our research on the second one.

8.3.1 Benign singularities

Our initial intention was to make just a small step – to construct examples of singularities which can be worked out. We call these *benign singularities*. The main property they have is that the metric tensor g_{ab} is smooth, the singular features occurring because $\det g = 0$ at the singular points. This allows the construction of the Christoffel symbols of the first kind

$$\Gamma_{abc} := \frac{1}{2} (\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}). \quad (8.22)$$

But $\det g = 0$ forbids the construction of the Christoffel symbols of the second kind $\Gamma^c_{ab} := g^{cs} \Gamma_{abs}$, which involves the reciprocal metric $g^{ab} := (g^{-1})^{ab}$. Consequently, the curvature has to be defined in terms of the Christoffel symbols of the first kind, and not of the second, as it is normal:

$$R_{abcd} = \partial_a \Gamma_{bcd} - \partial_b \Gamma_{acd} + \Gamma_{ac\bullet} \Gamma_{bd\bullet} - \Gamma_{bc\bullet} \Gamma_{ad\bullet}. \quad (8.23)$$

The symbol \bullet denote the contraction between covariant indices, and we adopted it because our contraction is more general than the usual one and the index notation would not be quite correct. Normally the contraction between covariant indices also requires the reciprocal metric g^{ab} , which becomes singular for $\det g = 0$. Luckily, the covariant contraction can be defined in a canonical and invariant way even in the case $\det g = 0$, provided that the tensor to be contracted satisfies certain conditions [67].

To see how, let's consider for starter two covectors $\omega, \tau \in T_p^* M$, where M is a semi-Riemannian manifold and $p \in M$. The contraction of the tensor $\omega \otimes \tau$ is given by $g^{st} \omega_s \tau_t$. This is defined by the metric g_{ab} , if $\det g \neq 0$. If $\det g = 0$, we can do this only

if there are two vectors $u, v \in T_p M$ so that $\omega_a = g_{as} u^s$ and $\tau_a = g_{as} v^s$. In this case, we define the contraction by

$$\omega_\bullet \tau_\bullet := g_{st} u^s v^t. \quad (8.24)$$

We denote by $T_p^\bullet M$ the subspace of $T_p^* M$ consisting of covectors (or 1-forms) of the form $\omega_a = (u^\flat)_a := g_{as} u^s$. We can extend this recipe to more general tensors from $\mathcal{T}_s^r(T_p M)$, provided that the components which we contract are from $T_p^\bullet M$. Note that we cannot use it to raise indices, since $u^\flat = (u + w)^\flat$ for any vector w so that $w^\flat = 0$. They form the kernel $\ker \flat$, which is zero if and only if $\det g = 0$.

From the above considerations we conclude that, to define the curvature as in (2.81), Γ_{abc} has to satisfy, at each point $p \in M$, the condition $\Gamma_{abs} w^s = 0$ for any vector $w^s \in T_p M$ which satisfies $g_{st} w^s v^t = 0$ for any vector $v \in T_p M$. Metrics satisfying this condition are named *radical-stationary*, and were studied by Kupeli for the case when the signature of the metric is constant [19, 20], and by the author for the general case [67, 73].

More details are given, in a manifestly invariant formulation, in [67], where we introduced also covariant derivatives for differential forms, and we defined the Riemann curvature tensor (2.81). The Riemann curvature tensor R_{abcd} can be defined in an invariant and canonical way, unlike R^a_{bcd} . Also, a simple condition ensured the smoothness of R_{abcd} , and such metrics and their singularities are named *semi-regular*. From the smoothness of R_{abcd} can be deduced in four dimensions the smoothness of $R_{ab} \det g$ and $R \det g$, and implicitly of the densitized Einstein tensor (5.6). This allows us to write a densitized version of Einstein's equation (5.10). This equation is equivalent to Einstein's equation outside the singularities, but remains smooth at the singularities. The smoothness condition may be too strong sometimes, but what is important is that the metric is radical-stationary.

A condition stronger than semi-regularity is that of *quasi-regularity*, which allows a smooth Ricci decomposition of the Riemann tensor, and leads to another extension of Einstein's equation – the *expanded Einstein equation*, which is tensorial (5.11).

Big-Bang singularities of this type satisfy Penrose's Weyl curvature hypothesis [59, 213] automatically [100].

A simple but central result in singular semi-Riemannian geometry is the generalization, given in [73], of *warped products* [3]. The warped products allow the construction of a large class of semi-regular and quasi-regular singularities. As a corollary, a warped product of (non-degenerate) semi-Riemannian manifolds is semi-regular, and in some cases quasi-regular. The warped product can be used to show that the singularity at the Big-Bang of the Friedmann-Lemaître-Robertson-Walker model is semi-regular [98], and quasi-regular [99].

8.3.2 Converting malign singularities into benign ones

There are solutions to Einstein's equation for which one or more components of the metric tensor are singular – let's call them *malign singularities*. For example, the stationary black holes apparently are of this type. But they can be tamed, by a change of coordinates.

To understand how, let's recall that initially it was considered that the Schwarzschild solution has a singularity on the event horizon, in addition to that in the “center” of the black hole. This opinion lasted until Eddington and Finkelstein proposed a coordinates in which the metric became manifestly finite on the event horizon [65, 66]. The metric was in fact only apparently singular on the event horizon.

For the Schwarzschild [101], Reissner-Nordström [96], and Kerr-Newman [97] black holes there are coordinate changes which make the metric analytic at the genuine singularity $r = 0$. For example, the Reissner-Nordström solution (7.48), characterizing the stationary black holes with electric charge q and mass m , the component $g_{tt} \rightarrow \infty$ as $r \rightarrow 0$. A coordinate change of the form (7.50) transforms the metric to the form (7.58). For $T > S \geq 1$, the singularity turns out to be benign [96].

These coordinates suggest the possibility that the standard coordinate systems obtained for the black holes are in fact singular, and the correct coordinates are analytic, like those we have found.

In the new coordinates for the Reissner-Nordström and Kerr-Newman metrics, the electromagnetic potential and the electromagnetic fields also become analytic, and they are finite even for $r = 0$.

The new coordinates allow the spacetime to be foliated. In the Reissner-Nordström case, this is ensured by the condition $T \geq 3S$. By continuously modifying the parameters characterizing the black holes (m , q , and a), we can construct models in which they appear and disappear by Hawking evaporation. Because of the existence of foliations, we can construct globally hyperbolic spacetimes containing very general singularities [102]. Thus, the singularities don't necessarily destroy information.

8.4 Dimensional reduction at singularities

Adopting dimensional reduction only to justify Quantum Gravity would be an ad-hoc solution. Fortunately, the dimensional reduction emerges from the very structure of singularities.

This suggests the possibility that the solution to the problem of singularities helps understanding the problem of quantization of gravity.

8.4.1 The dimension of the metric tensor

If the metric tensor g_{ab} is degenerate at a point $p \in M$, then the distance in a part of the directions vanishes (fig. 2.1). The vanishing directions are given by the isotropic vectors from the vector subspace $\ker(\flat) := T_p^\perp(M) \leq T_p M$, consisting of the tangent vectors which are orthogonal to $T_p M$. These directions can be eliminated if we take the quotient space

$$T_{p\bullet}M := \frac{T_p M}{T_p^\perp(M)} \quad (8.25)$$

whose dimension is equal to the rank of the metric at p :

$$\dim T_{p\bullet}M = \text{rank } g_p. \quad (8.26)$$

Then,

$$T_p^\bullet M := \flat(T_p M) \quad (8.27)$$

has the same dimension as $T_{p\bullet}M$, and in fact they are related by the isomorphism

$$\flat : T_{p\bullet}M \rightarrow T_p^\bullet M \quad (8.28)$$

induced by the morphism

$$\flat : T_p M \rightarrow T_p^\bullet M \leq T_p^* M. \quad (8.29)$$

We see that, because the metric tensor is degenerate, it can be reduced to a metric tensor on the space $T_{p\bullet}M$, and its inverse is a metric tensor on $T_p^\bullet M$. The dimension is actually given by the rank of the metric, which can be viewed as the dimension of the metric tensor.

8.4.2 Metric dimension *vs.* topological dimension

Let's consider a vector space V of dimension D . A symmetric bilinear form g on V defines a scalar product, or a metric. If g is degenerate, $\text{rank } g < D$. The distance vanishes in the directions from V^\perp , and the geometric dimension is given by the rank of the metric. It is not necessarily equal to the vector space dimension D .

Let's consider now a semi-Riemannian manifold (M, g) . Even if the rank of the metric is at some points lower than $D = \dim M$ (when the metric becomes degenerate), the topological dimension of the manifold remains D .

From mathematical viewpoint, in Differential Geometry there are three layers: the *topological structure*, the *differential structure* and the *geometric structure*. The topological structure on the set M is given by an atlas of local charts mapping an open set from M with one from \mathbb{R}^D , so that the transition maps are continuous. If the transition maps are differentiable, M becomes a differentiable manifold. If we add a metric tensor on M , we obtain a geometric structure. The topological dimension of M is the dimension $D = \dim \mathbb{R}^D$ of the vector space used in the charts of the atlas. The *metric dimension*, or the *geometric dimension* is given by the rank of the metric, and is allowed to be at most equal to the topological dimension.

In the case when the metric is degenerate of constant signature (k, l, m) and is radical-stationary, a theorem of Kupeli [19] shows that the manifold (M, g) is locally isomorphic to a direct product manifold $P \times_0 N$ between a k -dimensional manifold N (without a metric, or with metric equal to 0) and a (non-degenerate) semi-Riemannian manifold P of signature (l, m) . Hence, from the viewpoint of the metric g on M , at any point $p \in M$ the $k = D - \text{rank } g$ dimensions associated with the degenerate directions can be ignored locally. We can thus identify the D -dimensional manifold M around the point p , with the rank g -dimensional manifold P . The manifold (M, g) looks locally like a lower-dimensional manifold (P, h) . This situation is analogous to that of the gauge degrees of freedom.

If the metric is radical-stationary and has variable signature, the manifold (M, g) can be identified piecewisely with lower-dimensional manifolds (P, h) . The information contained in the metric g of the manifold M can be obtained by pull-back from that of a metric on a manifold (P, h) with variable topological dimension.

This establishes the connection with the topological dimensional reduction proposed and studied by D.V. Shirkov and P. Fiziev (see section §8.2.1.3). Our approach leads to a conclusion which is very close to the following observation from [124]:

dimensional reduction of the physical space in general relativity (GR) can be regarded as an unrealized and as yet untapped consequence of Einstein's equations (EEqs) themselves which takes place around singular points of their solutions.

It is not clear at this point to what extent the geometric reduction of radical-stationary semi-Riemannian manifolds is equivalent to the semi-Riemannian manifolds with variable topological dimension. But what we can say is that in GR there are other fields to be considered in addition to the metric tensor. The information contained in those fields will be, in general, lost by the topological reduction induced by the metric dimensional reduction. On the other hand, in order to admit smooth covariant contractions and smooth covariant derivatives (as defined in [67] for differential forms and tensors with covariant indices), the fields are required to ignore to some extent the degenerate directions. But even under these conditions, they are more general than the fields defined on manifolds of lower topological dimension, and we cannot recover the former from the latter ones by pull-back.

Keeping the points topologically distinct, even though the distance induced by the metric vanishes between them, provides more generality than allowing the topological dimensional reduction.

One important reason to avoid making the topological identification due to the dimensional reduction is that variable topological dimension is not compatible with the foliation of the spacetime in spacelike hypersurfaces. This kind of foliation is important for global hyperbolicity and for avoiding the information loss [102].

8.4.3 Metric dimensional reduction and the Weyl tensor

At quasi-regular singularities the Ricci decomposition is smooth. According to a theorem we proved in [100], the Weyl curvature vanishes at quasi-regular singularities. The reason is that $\dim T_p^\bullet M < D$, hence the dimension of the Riemann curvature tensor R_{abcd} is reduced. For $D = 4$, $\dim T_p^\bullet M \leq 3$, consequently any tensor having the algebraic properties of the Weyl tensor vanishes. Because the Ricci decomposition is smooth, in particular the Weyl curvature tensor C_{abcd} is smooth, and this means that around the singularity the Weyl tensor remains small.

An example of quasi-regular singularity is the Schwarzschild black hole [84], which can be used as a classical model for neutral spinless particles. At this time it is not clear whether the singularities of the stationary charged and rotating black holes are quasi-regular, but they are analytic, the geometric dimensional reduction occurs, and the electromagnetic potential and its field are analytic and finite at $r = 0$ [96, 97].

The vanishing of the Weyl curvature tensor implies that the local degrees of freedom, *i.e.* the gravitational waves for GR and the gravitons for QG, are absent [122]. Because of

continuity, as approaching a quasi-regular singularity, the contribution of the gravitons vanishes, and this diminishes the renormalizability problems of QG.

8.4.4 Lorentz invariance and metric dimensional reduction

When the metric of a spacetime becomes degenerate, it is no longer Lorentzian. The group of transformations which is associated to the metric can't be a Lorentz group, since the metric is degenerate. The role of the Lorentz group is taken by a larger group – the *Barbilian group* [214]. But if the metric is radical-stationary, the Barbilian group can be reduced to a subgroup, which is a Lorentz group of lower dimension.

We can say, because of this, that the condition to be radical-stationary is the mathematical expression of the Lorentz invariance, in the case when the metric is allowed to be degenerate.

The spacetimes with this kind of metric satisfy the Lorentz invariance so long as the metric is non-degenerate, and when it becomes degenerate, the Lorentz invariance is maintained, but for lower dimensions. One should mention here that this is the best we can do, if we want to include the singularities in the spacetime. Having the full 4-dimensional Lorentz invariance at singularities is not possible, because of the very definition of singularities.

By comparison, other approaches to QG had to give up Lorentz invariance even outside the singularities. It is the case of loop quantum gravity and Hořava-Lifschitz gravity for example, where it is hoped that the four-dimensional Lorentz invariance emerges at large scales.

8.4.5 Particles lose two dimensions

As described in [96, 97] and in §8.3.2, a charged black hole of type Reissner-Nordström or Kerr-Newman can be described by an analytic metric, and the electromagnetic potential and its field are also analytic and remain finite at $r = 0$. From the viewpoint of GR, charged particles are such black holes. Also this applies to other types of gauge fields, such as the Yang-Mills fields².

As we can see from (7.58), in our coordinates, at $\rho = 0$ (which is equivalent to $r = 0$), the metric loses two dimensions, those corresponding to coordinates ρ and τ . Apparently

²Of course, the relativistic effects should be complemented by the quantum properties to get a complete description of a physical particle, and we don't have yet such a model.

the metric on the sphere vanishes too, but this only reflects that the warped product describing spherically symmetric solutions to Einstein's equation involves all the concentric spheres, down to $r = 0$.

Metric dimensional reduction occurs similarly for the Schwarzschild (describing neutral particles) and the more general Kerr-Newman (describing charged or neutral, spinning or not particles) cases.

The fact that two dimensions are lost, and that the gauge potential and fields remain finite at the singularity, are expected to have important impact to field quantization.

One of the major problems of electrodynamics is the fact that the particle's potential and field become infinite as approaching $r = 0$. This problem turns out to be removed, by employing our non-singular coordinate system. This sets as one priority in the future developments of our program to see exactly what happens in the perturbative expansions and in the renormalization group analysis.

8.4.6 Particles and spacetime anisotropy

The metric (7.58) admits a foliation in spacelike hypersurfaces only for $T \geq 3S$ in (7.50) [96]. This condition allows the coordinates to be compatible with the distinction between space and time. But it leads to an anisotropy between space and time, which is manifest when passing to the old Reissner-Nordström coordinates (t, r) . A rescaling in the coordinates (τ, ρ) is isotropic, of course in the coordinates (τ, ρ) , but the coordinates (t, r) are rescaled anisotropically, due to (7.50).

The diffeomorphism invariance should be considered valid in coordinates (τ, ρ) , but not in the singular coordinates (t, r) .

This anisotropic scaling invariance is similar to that which P. Hořava managed to obtain by modifying the Lagrangian of GR in [196]. His analysis shows that the anisotropy would lead to the correct dimension for the Newton constant (see §8.2.2.3 and [196]). Apparently, the anisotropy we obtained in [96] is not equivalent to that from Hořava-Lifschitz gravity, because ours follows from standard GR considerations, while that of Hořava from modifications of the Einstein-Hilbert Lagrangian (and implicitly of Einstein's equation).

In our proposal, the metric is still the fundamental field. The very degeneracy of the metric imposes conditions on the foliation at the singularity. There is no need for other structure to define the foliation, as it is in Hořava-Lifschitz gravity. In our approach there is no need to impose in the IR limit the recovery of standard GR, since we start from standard GR outside the singularities, and extend it in the singularities.

8.4.7 The measure in the action integral

When the metric becomes degenerate, its determinant vanishes. Consequently, the volume form

$$d_{vol} := \sqrt{-\det g} dx^0 \wedge \dots \wedge dx^{D-1} = \sqrt{-\det g} dx^D \quad (8.30)$$

tends to 0 as approaching the degenerate singularities (in non-singular coordinates, in which the metric is smooth).

The action principle is given by

$$S = \int_{\mathcal{M}} d_{vol}(x) \mathcal{L} \quad (8.31)$$

If the metric is diagonal in the coordinates (x^μ) , then

$$d_{vol}(x) = \prod_{\mu=0}^{D-1} \sqrt{|g_{\mu\mu}(x)|} dx^D. \quad (8.32)$$

This is in fact similar to the measure studied by G. Calcagni [154, 155]. This is obvious if we take in the Lebesgue measure from §8.2.1.2, (8.1), the following weights:

$$f_{(\mu)}(x) = \sqrt{|g_{\mu\mu}(x)|}. \quad (8.33)$$

This identification makes much of the analysis developed in [154, 155] for QFT be a consequence of the fact that the metric may be degenerate.

In terms of $v(x) = \prod_{\mu=0}^{D-1} f_{(\mu)}(x)$, we have

$$v = \sqrt{-\det g}, \quad (8.34)$$

so that the measure becomes

$$d\rho(x) = \sqrt{-\det g} dx^D, \quad (8.35)$$

which is just the standard measure from General Relativity, except that in our framework it is allowed to vanish.

The justification for changing the measure in G. Calcagni's proposal comes in his theory from the hypothesis that the spacetime is fractal, the Hausdorff dimension changing with the scale. The fractal nature manifests in the usage of fractional calculus. He uses a general Lebesgue-Stieltjes measure, independent of GR, and applies it to flat spacetime QFT, as well as to QG [154–156].

The action (8.1) refers to the Special Relativistic QFT, but the identification (8.33) we proposed replaces the weights f_μ with the square root of the metric components, which we allowed to tend to zero. This suggests that GR comes to rescue QFT at high energies, by reducing the dimension to two. For GR, Calcagni applies the same recipe he used in Special Relativistic QFT: proposes that the weight v multiplies $\sqrt{-\det g}$ [154]. This would of course lead to a modified Einstein equation, and modified GR. Our approach sticks to the standard GR, the only difference being in allowing the metric to become degenerate.

In our solution, the measure is simply due to the degeneracy of the metric. Our claim is that GR itself provides the measure and tames the infinities of QFT.

8.4.8 Does dimension vary with scale?

So far we provided arguments that the singularities characterized by the degeneracy of the metric explain the geometric dimensional reduction. This dimensional reduction has many common features with the dimensional reduction expected in other proposals in the literature. In addition, it is not invented with the problem of quantization in mind, but it is a consequence of our approach to the singularity problem, which in turn fits naturally in classical GR.

Yet, the hardest part remains to be done. The geometric dimensional reduction we propose becomes manifest as the distance to a singularity becomes smaller. But the dimensional reduction needed in QFT and QG seems to have nothing to do with the distance to a singularity. It is required just to depend on the scale.

The precise ways in which the geometric dimensional reduction we proposed impacts QFT and GR need to be analyzed in more depth. At this point, we will give only a “qualitative” (*i.e.* “handwaving”) justification.

The higher order Feynman diagrams involve a larger number of particles. This means that in the same region of space there will be more particles, which we will consider to be benign singularities. Because of this, the metric will have, in average, smaller determinant. Recall that for benign singularities the determinant of the metric tends to 0 as the distance to the singularity decreases, and having a higher number of singularities in the same region reduces the average of the metric’s determinant (fig. 8.1). Thus, in the high energy limit, the measure dimensional reduction will become more and more present in the integrals.

Let’s discuss a bit the diffeomorphism invariance (general covariance) of the above argument. It is clear that both $\det g$ and other relevant quantities such as the components

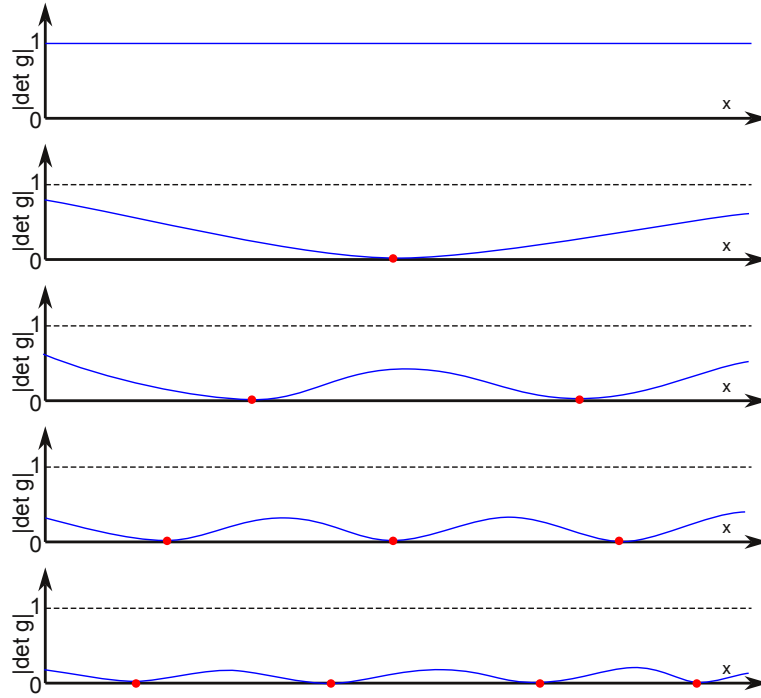


FIGURE 8.1: Schematic picture illustrating how we expect that the metric's average determinant decreases as the number of singularities (*i.e.* particles) in the region increases. The red dots represent the singularities, and the blue line represents $|\det g|$.

of the Weyl tensor C_{abcd} depend on the coordinates. The singularities describing the particles provide some constraints that $\det g = 0$, and $C_{abcd} = 0$ for quasi-regular singularities, along the universe lines describing the particles. These constraints are invariant under local diffeomorphisms. It is true that changing the coordinates one can make these quantities increase away from the singularities, but the constraints are invariant. Also, $\det g$ is present in the action, and the action integral is invariant.

In conclusion, *the main conjecture* is that, although the dimensional reduction happens at singularities which represent the particles, when many particles are present, the measure dimensionality – present through $\det g$ – is reduced in average. We claim that this acts like a regulator. Let's name this *the hypothesis of average dimensional reduction*.

If this hypothesis is true, then the fact that the average metric determinant changes with the scale is not an absolute law, but merely an “accident”. If the real reason of change is the way the singularities are distributed in a given region, then this may have observable consequences at larger scales too. But at larger scales, the predictions of GR we expected so far seem to be confirmed with high degree of accuracy, yet nothing like this has been found. On the other hand, let's not forget that most tests of GR concern phenomena involving a small number of bodies which are separated by large regions of vacuum. We need to derive and test predictions for the case when matter is

distributed more homogeneously, as the galaxies appear at large scale. And we know that at large scale, the galaxies don't seem to fit what we presently expect to be the behavior predicted by GR, as we understand it so far. It would be interesting to check if some of the discrepancies usually attributed to *dark matter* can in fact be explained as large scale manifestations of the average dimensional reduction. This may be supported by the idea that the galaxies rotate as if they were $(1 + 2)$ -dimensional [215].

8.5 Conclusions

We reviewed some of the hints indicating that if a dimensional reduction would take place at small scales, then some major problems concerning the quantization of gravity, but also of other fields, would go away. Some hints refer to the dimension involved in calculations, others to the geometric and topological dimensions, and others to the spectral dimension.

We advocated here the position that the approach to singularities introduced and developed in [67, 73, 84, 96–102], leading to a (geo)metric dimensional reduction, also lays a foundation for the quantization of gravity. This position is supported by the strong connections between the metric dimensional reduction and the other kinds of dimensional reductions, reviewed in this chapter.

This is just a small step; many questions remain open, and much work remains to be done.

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